Solutions

1. $\lim_{x\to -\frac{2}{3}} \frac{9x^3-7x-2}{9|x^2-1|-5}$

The numerator and denominator vanish as $x \to -\frac{2}{3}$, so each is divisible by $x+\frac{2}{3}$, or $3x+2$, provided x is sufficiently near $-\frac{2}{3}$. By inspection, $9x^3-7x-2=(3x+2)(3x^2-2x-1)$. Also, $|x^2-1| = 1-x^2$ if $-1<x<1$, in which case $9|x^2-1|-5 = 4-9x^2 = (2-3x)(2+3x)$. Therefore, the limit is equal to

$$\lim_{x\to -\frac{2}{3}} \frac{3x^2-2x-1}{2-3x} = \frac{\frac{4}{3}+\frac{4}{3}-1}{2+2} = \frac{5}{12},$$

by independence and direct substitution.

2. $\lim_{y\to 0} \frac{2\tan(3y)-\sin(y)}{3y+2\sin(2y)}$

Multiplying and dividing by $1/y$ and revising gives

$$\lim_{y\to 0} \frac{2\tan(3y)-\sin(y)}{3y+2\sin(2y)} = \lim_{y\to 0} \frac{\frac{\sin(3y)}{3y}\cdot\frac{6}{\cos(3y)} - \frac{\sin(y)}{y}}{3+4\cdot\frac{\sin(2y)}{2y}} = \frac{6-1}{3+4} = \frac{5}{7}.$$

3. $\lim_{x\to -2} \frac{2x+1+\sqrt{5-2x}}{x^3+8}$

The numerator and denominator each vanish as $x\to 4$, and the denominator is a sum of cubes, giving $x^3+8=(x+2)(x^2-2x+4$. If $a=2x+1)$ and $b=\sqrt{5-2x}$, then $a\to -3$ and $b\to 3$ as $x\to -2$, and the factorization of a difference of squares gives

$$(a+b)(a-b) = a^2-b^2 = 4x^2+4x+1-(5-2x) = 4x^2+6x-4 = 2(x+2)(2x-1)$$

or

$$a+b = \frac{2(x+2)(2x-1)}{a-b}.$$

Therefore, the limit is equal to

$$\lim_{x\to -2} \frac{2(2x-1)}{(x^2-2x+4)(a-b)} = \frac{2(-5)}{(4+4+4)(-3-3)} = \frac{5}{36},$$

by independence and direct substitution.

4. $\lim_{x\to -\infty} \left\{2x+1+\sqrt{3-5x+4x^2}\right\}$

As $x\to -\infty$ the apparently dominant terms $2x$ and $\sqrt{4x^2}=-2x$ cancel. Rationalizing the numerator and then inspecting dominant terms gives

$$\frac{(2x+1)^2-(3-5x+4x^2)}{2x+1-\sqrt{3-5x+4x^2}} = \frac{9x-2}{2x+1-\sqrt{3-5x+4x^2}} \to \frac{9}{2-(-2)} = \frac{9}{4}$$

as $x\to -\infty$.

5. $\lim_{x\to 0} \frac{1-\cos(6x)}{x\tan(5x)}$

Since $1-\cos(6x) = 2\sin^2(3x)$, the limit is equal to

$$\lim_{x\to 0}\left\{2\cdot\left(\frac{\sin(3x)}{3x}\right)^2\cdot\left(\frac{5x}{\sin(5x)}\right)\cdot\frac{9\cos(5x)}{5}\right\} = 2\cdot 1^2\cdot 1\cdot\frac{9}{5} = \frac{18}{5}.$$

6. $\lim_{\vartheta\to \frac{4}{3}\pi^-} \frac{\tan(2\vartheta)}{2\cos(\vartheta)+1}$

As $\vartheta \to \frac{4}{3}\pi^-$, $\tan(2\vartheta)\to\tan\left(\frac{8}{3}\pi\right) = -\sqrt{3}$ and $\cos(\vartheta)\to -\frac{1}{2}^-$, so $2\cos(\vartheta)+1\to 0^-$. Therefore, the limit is ∞.

7. $\lim_{x\to 7} \frac{\frac{x}{2x-5}-\frac{7}{x+2}}{\sqrt[3]{4x-1}-3}$

First of all, since $x(x+2)-7(2x-5) = x^2-12x+35 = (x-7)(x-5)$, it follows that

$$\frac{x}{2x-5}-\frac{7}{x+2} = \frac{(x-7)(x-5)}{(2x-5)(x+2)}.$$

Next, $a=\sqrt{4x-1}$ then $a\to 3$ as $x\to 7$, $a^3-27 = 4x-28 = 4(x-7)$ and the factorization of a difference of cubes gives

$$\frac{1}{a-3} = \frac{a^2+3a+9}{a^3-27} = \frac{a^2+3a+9}{4(x-7)}.$$

Therefore, the limit is equal to

$$\lim_{x\to 7} \frac{(x-5)(a^2+3a+9))}{4(2x-5)(x+2)} = \frac{2(9+9+9)}{4\cdot 9\cdot 9} = \frac{1}{6},$$

by independence and direct substitution.

8. $\lim_{x\to 5} \frac{\sqrt{2x-1}-3}{2-\sqrt{x-1}}$

If $a=\sqrt{2x-1}$ then $a^2-9 = 2x-10 = 2(x-5)$, if $b=\sqrt{x-1}$, then $4-b^2 = 5-x$, and the factorization of a difference of squares gives

$$a-3 = \frac{a^2-9}{a+3} = \frac{2(x-5)}{a+3} \quad\text{and}\quad \frac{1}{2-b} = \frac{2+b}{4-b^2} = \frac{2+b}{5-x}.$$

Since $a\to 3$ and $b\to 2$ as $x\to 5$, the limit is equal to

$$\lim_{x\to 5} \frac{-2(2+b)}{a+3} = \frac{-2\cdot 4}{6} = -\frac{4}{3},$$

by independence and direct substitution.

9. Find all values of a and b, if any, such that $\lim_{x\to 2} \frac{\sqrt{ax+b}-3}{2x^4-9x^2+4} = \frac{1}{24}$.

The denominator vanishes as $x\to 2$, and factorizing gives

$$2x^4-9x^2+4 = (x-2)(2x^3+4x^2-x-2).$$

The limit is undefined unless the numerator also vanishes as $x\to 2$, which gives $\sqrt{2a+b}=3$, or $b-9=-2a$. If $c=\sqrt{ax+b}$ then $c^2-9 = ax+b-9 = ax-2a = a(x-2)$, and the factorization of a difference of squares gives

$$c-3 = \frac{c^2-9}{c+3} = \frac{a(x-2)}{c+3}.$$

Therefore, the limit is equal to

$$\lim_{x\to 2} \frac{a}{(c+3)(2x^3+4x^2-x+2)} = \frac{a}{6(16+16-2-2)} = \frac{a}{168},$$

by independence and direct substitution. So the limit is equal to $\frac{1}{24}$ if, and only if, $a=\frac{168}{24}=7$, and $b=9-2a=9-14=-5$.