Three exercises on tangent lines, with solutions

Note. — In your solutions show all relevant calculations and simplify numerical values as much as possible. It is sufficient to give an equation in point-slope form. It is not necessary, and it is not worth any additional credit, to write an equation in the form $y = ax + b$.

Exercise 1. — Find an equation of the line tangent to the graph of $y = (x - 4\sqrt{x})(x - 12\sqrt[3]{x})$ at the point with x coordinate 64.

Solution to exercise 1. — If $y = (x - 4\sqrt{x})(x - 12\sqrt[3]{x}) = x^2 - 4x^{3/2} - 12x^{4/3} + 48x^{5/6}$ then

$$y\Big|_{x=64} = (64 - 32)(64 - 48) = 32 \cdot 16 = 512$$

and

$$\frac{dy}{dx}\Big|_{x=64} = \left(2x - 6x^{1/2} - 16x^{1/3} + 40x^{-1/6}\right)\Big|_{x=64} = 128 - 48 - 64 + 20 = 36.$$

So the equation of the line tangent to the curve where $x = 64$ is defined by $y = 512 + 36(x - 64)$.

Exercise 2. — Write an equation of the tangent line to the curve defined by $y = 2x^3 - 3x^2 + 5$ at the point where $x = 2 - \sqrt{2}$.

Solution to exercise 2. — First of all,

$$y\Big|_{x=2-\sqrt{2}} = 2(2 - \sqrt{2})^3 - 3(2 - \sqrt{2})^2 + 5 = 2(20 - 14\sqrt{2}) - 3(6 - 4\sqrt{2}) + 5 = 27 - 16\sqrt{2}.$$

Next,

$$\frac{dy}{dx}\Big|_{x=2-\sqrt{2}} = (6x^2 - 6x)\Big|_{x=2-\sqrt{2}} = 6x(x - 1)\Big|_{x=2-\sqrt{2}} = 6(2 - \sqrt{2})(1 - \sqrt{2}) = 6(4 - 3\sqrt{2}).$$

So the tangent line is defined by $y = 27 - 16\sqrt{2} + 6(4 - 3\sqrt{2})(x - 2 + \sqrt{2})$.

Exercise 3. — Let

$$f(x) = \frac{x - 18\sqrt{x} - 27}{\sqrt[3]{x}}.$$

Find an equation of the line tangent to the graph of f at the point with x coordinate 27.

Solution to exercise 3. — First of all

$$f(27) = \frac{27 - 18\sqrt{27} - 27}{\sqrt[3]{27}} = -18\sqrt{3}.$$

Next, $f(x) = x^{2/3} - 18x^{1/6} - 27x^{-1/3}$, so

$$f'(27) = \tfrac{2}{3} \cdot 27^{-1/3} - 3 \cdot 27^{-5/6} + 9 \cdot 27^{-4/3} = \tfrac{2}{9} - \tfrac{1}{9}\sqrt{3} + \tfrac{1}{9} = \tfrac{1}{3} - \tfrac{1}{9}\sqrt{3}.$$

So the tangent line is defined by $y = \left(\frac{1}{3} - \frac{1}{9}\sqrt{3}\right)(x - 27) - 18\sqrt{3}$.