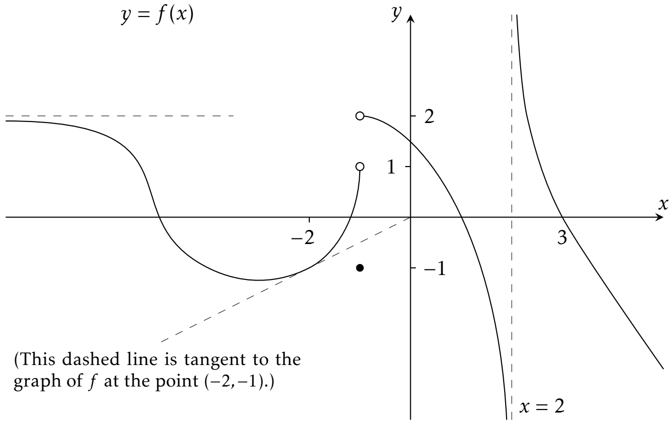
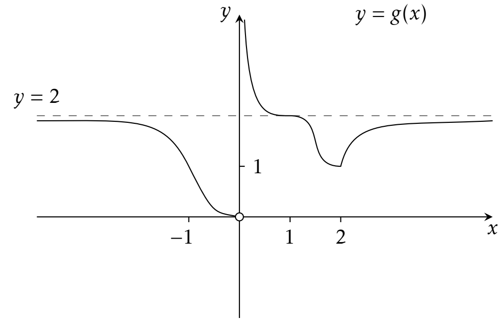
Review exercises

Exercise 1. Below are graphs of two functions f and g. Assume that the curves continue as suggested naturally by the pictures.





Referring to these graphs, evaluate each of the following. Give the left and right limits, and use the symbols ∞, $-\infty$ as appropriate.

a. $\displaystyle\lim_{x\to 0}\frac{f(x-1)}{f(x+1)}$ b. $\displaystyle\lim_{x\to 0} f\big(1-g(1/x)\big)$ c. $\displaystyle\lim_{x\to -1}\big\{f(x)g(x)\big\}$ d. $\displaystyle\lim_{x\to 0}\big\{f(x-1)g(-x^2)\big\}$ e. $\displaystyle\lim_{x\to 0^-}\frac{f(x)}{g(x)}$

f. $\displaystyle\lim_{x\to\infty}(f\circ g)(x)$ g. $\displaystyle\lim_{x\to 0}(g\circ g)(x)$ h. $\displaystyle\lim_{x\to -1}\big\{f(x)+(g\circ f)(x)\big\}$ i. $\displaystyle\lim_{x\to 2}\frac{g(x)}{f(x)}$ j. $\displaystyle\lim_{x\to -2}\frac{f(x)+1}{x+1+\sqrt{3x^2+5x-1}}$

Exercise 2. Evaluate each of the following limits. Give the left and right limits, and use the symbols $-\infty$, ∞ as appropriate.

a. $\displaystyle\lim_{x\to\frac{3}{2}}\frac{2x^2+x-7}{4x^2-12x+9}$ b. $\displaystyle\lim_{\vartheta\to\frac{5}{3}\pi}\frac{\sin\vartheta}{1-2\cos\vartheta}$ c. $\displaystyle\lim_{t\to 0}\frac{\sin t}{t^{-1}+\sqrt{1+t^{-2}}}$ d. $\displaystyle\lim_{t\to 0}\frac{\cos(1/t)}{1-e^{-1/t}}$ e. $\displaystyle\lim_{x\to 5}\frac{\frac{1}{x-7}+\frac{x-3}{4}}{3-\sqrt{2x-1}}$

f. $\displaystyle\lim_{t\to\frac{1}{2}}\frac{2t^2+t-1}{2t^3+3t^2-1}$ g. $\displaystyle\lim_{x\to 0}\frac{x\sin 2x}{1-\cos 3x}$ h. $\displaystyle\lim_{x\to 3^-}\tan\left(\pi\cos\left(\frac{2\pi}{x}\right)\right)$ i. $\displaystyle\lim_{x\to-\frac{1}{3}}\frac{(1-3x)^{3/2}-2\sqrt{2}}{9x^2-1}$ j. $\displaystyle\lim_{t\to -1}\left\{\frac{3}{t^2-t-2}-\frac{t}{t+1}\right\}$

k. $\displaystyle\lim_{\varphi\to\infty}\varphi\tan^2\left(\frac{\pi}{\sqrt{\varphi}}\right)$ l. $\displaystyle\lim_{r\to-\infty}\frac{r+\sqrt{2r^2+2r-3}}{9+3r}$ m. $\displaystyle\lim_{x\to -1}\frac{x^2-1}{|2x^2+x-1|}$ n. $\displaystyle\lim_{x\to -3}\frac{5x^2-|14x-3|}{2x^3+|15x-9|}$ o. $\displaystyle\lim_{x\to\infty}\left\{2\sqrt{x+\sqrt{3x}}-\sqrt{4x-1}\right\}$

Exercise 3. Find and classify the discontinuities of the function f, defined by

$$f(x)=\begin{cases}2x+3 & \text{if } x\leqslant -1,\\ \dfrac{2x^2-x+1}{(x-1)^2} & \text{if } -1<x<1, \text{ and}\\ \dfrac{x-2}{x^2+3x-10} & \text{if } 1<x \text{ and } x\neq 2.\end{cases}$$

To earn full credit, write a complete and well-organized solution.

Exercise 4. Find all values of a that make h continuous on $\mathbb{R}$, where

$$h(x)=\begin{cases}3x^2+a^3 & \text{if } x<2, \text{ and}\\ ax^2+a^2x+a^2 & \text{if } 2\leqslant x.\end{cases}$$

To earn full credit, write a complete and well-organized solution.

Exercise 5. Find all values of a and b so that the function f, defined by

$$f(x)=\begin{cases}x^2+6x-18 & \text{if } x<b,\\ a & \text{if } x=b \text{ and}\\ 5x-6 & \text{if } x>b,\end{cases}$$

is continuous on $\mathbb{R}$. To earn full credit, write a complete and well-organized solution, and simplify your final answers as much as possible.

Exercise 6. Given that

$$x-\tfrac{1}{6}x^3<\sin x<x-\tfrac{1}{6}x^3+\tfrac{1}{120}x^5,$$

at least if $0<x<1$, evaluate the limit

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}.$$

For full marks, write a complete and well-organized solution.

Exercise 7. The function f is the ratio of two cubic polynomial functions. The graph of f has x-intercepts at $\left(-\frac{5}{3},0\right)$ and $(1,0)$, vertical asymptotes defined by $x=\frac{2}{7}$ and $x=-5$, the horizontal asymptote of its graph is defined by $y=-4$, and f has a removable discontinuity at 4.

a. Give an formula for $f(x)$.

b. What is the y-intercept of the graph of f?

c. Evaluate $\displaystyle\lim_{x\to 4} f(x)$.

Exercise 8. Give the intervals of continuity, and all asymptotes of the graph, of each function.

a. $f(x)=[\![x]\!]+[\![-x]\!]$ b. $f(x)=\dfrac{3e^x+2}{e^x-1}$

c. $g(x)=\dfrac{5x^4+20x^3+23x^2+12x+12}{(3x+2)^3(x+2)}$

Exercise 9. Mark each statement as true or false. Justify your conclusions briefly.

a. If $\displaystyle\lim_{x\to a} f(x)$ is ∞, or $-\infty$, then $\displaystyle\lim_{x\to a}\{1/f(x)\}=0$.

b. If $\displaystyle\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)$ then f is continuous at a.

c. If f is continuous at a then $\displaystyle\lim_{x\to a} f(x)$ exists.

d. If $\displaystyle\lim_{x\to 0^+} f(x)=0$ then $\displaystyle\lim_{x\to 0^+}\{1/f(x)\}$ is ∞, or $-\infty$.

e. If $\displaystyle\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}=\ell$, then f is continuous at a.

f. If $\displaystyle\lim_{x\to a}\left\{\big(f(x)\big)^2\right\}=\ell$ then $\displaystyle\lim_{x\to a} f(x)=\sqrt{\ell}$.

Exercise 10. Give an expression defining a function f such that

$$\lim_{x\to 0} f(x)=\infty,\qquad \lim_{x\to\pm\infty} f(x)=0$$

and f has infinitely many discontinuities. Justify all claims carefully.

AN OLD TEST 1

Marking notes

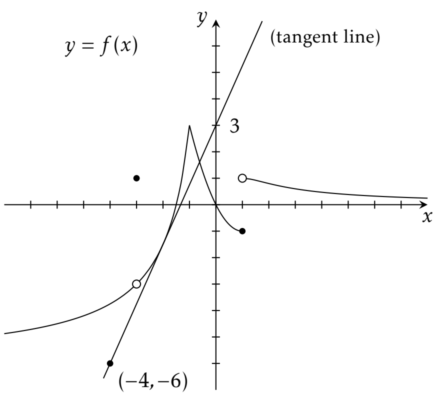
You might see the following symbols when your your solutions have been graded and returned.

- **N :** You have used incorrect *notation*; *e.g.*, absence or misuse of the limit symbol, or of the equality symbol.
- **G :** You have written nonsense; *e.g.*, "$1/\infty = 0$" or "$2/0 = \pm\infty$." (The G stands for *gibberish*.)
- **J :** An essential point of *justification* is missing from your solution.

The use of calculators is neither needed nor permitted.

Questions

Question 1. Below you are given the graph of a function f, with unit lengths marked along the coordinate axes. and the line tangent to the graph at the point where $x = -2$. Unit lengths are marked on the coordinate axes.



Answer each of the following. Include left and right limits where necessary, and use the symbols ∞ and $-\infty$ as appropriate. For full marks, explain your answers precisely and concisely.

a. Evaluate:

i. $\displaystyle\lim_{x\to 1}\frac{f(x)}{f(x-1)}$; ii. $\displaystyle\lim_{x\to 0} f(2-\cos x)$; iii. $f(-3)$;

iv. $\displaystyle\lim_{x\to 0^-} f\big(1/f(x)\big)$; v. $\displaystyle\lim_{x\to -2}\frac{f(x)-f(2)}{x+2}$

b. On which intervals is f continuous? (Give the largest intervals.)

c. Identify and classify all discontinuities of f.

Question 2. Evaluate each of the following limits. To earn full credit, write complete and well-organized solutions.

a. $\displaystyle\lim_{r\to 2}\frac{\sqrt{5r-1}-\sqrt{2r^2+1}}{r-2}$ b. $\displaystyle\lim_{t\to 1}\left\{\frac{1}{t^2-3t+2}+\frac{1}{t-1}\right\}$

Question 3. Evaluate each of the following limits. Include the left and right limits, and use the symbols ∞ and $-\infty$, as appropriate. For full marks, write complete and well-organized solutions.

a. $\displaystyle\lim_{x\to\frac{1}{2}}\frac{|2x-1|}{1-4x+4x^2}$ b. $\displaystyle\lim_{t\to 0}\frac{\sin t}{t^{-1}+\sqrt{1+t^{-2}}}$

Question 4. Evaluate each of the following limits. For full marks, write complete and well-organized solutions.

a. $\displaystyle\lim_{x\to-\infty} x\sin\left(\frac{5}{x}\right)$ b. $\displaystyle\lim_{x\to\infty}\left\{\sqrt{3x^2-2x+7}-\sqrt{3x^2-5x+9}\right\}$

c. $\displaystyle\lim_{x\to\infty}\left\{2x-\ln(e^{2x-3}+1)\right\}$ d. $\displaystyle\lim_{x\to\infty}\left\{\sqrt[3]{5x^3-3x^2+1}-x\sqrt[3]{5}\right\}$

Question 5. Evaluate each of the following limits. For full marks, write complete and well-organized solutions. In Part c, n is a positive integer.

a. $\displaystyle\lim_{x\to-2}\frac{3x^3+7x^2-4}{3x^2+5x-2}$ b. $\displaystyle\lim_{\vartheta\to 0}\frac{1-\cos(1-\cos\pi\vartheta)}{\vartheta^2(1-\cos 3\vartheta)}$

c. $\displaystyle\lim_{x\to 1}\frac{x^n-nx+n-1}{(x-1)^2}$ d. $\displaystyle\lim_{t\to x}\frac{\dfrac{t}{\sqrt{t^2+4}}-\dfrac{x}{\sqrt{x^2+4}}}{t-x}$

Question 6. Find all (vertical, horizontal and oblique) asymptotes of each function.

a. $f(x)=\dfrac{x+\sqrt{3x^2-x+1}}{2x-3}$ b. $g(x)=\dfrac{1-\cos x}{x\sin x}$

c. $h(x)=\dfrac{2x^3+3x^2-1}{4x^3-3x+1}$ d. $h(x)=\dfrac{x^2\cos\left(\frac{3}{\sqrt{x}}\right)}{2x-5}$

Question 7. The function f is defined by

$$f(x)=\begin{cases}\dfrac{x}{x+2} & \text{if } x<1 \text{ and } x\neq -2,\\[2mm] \cos\left(\dfrac{\pi}{x}\right) & \text{if } 1\leqslant x<3,\\[2mm] \dfrac{1}{x-1} & \text{if } 3\leqslant x.\end{cases}$$

Determine the numbers at which f is not continuous and the type of each discontinuity. Write a complete and well-organized solution.

Question 8. Find all values of a for which the function f, defined by

$$f(x)=\begin{cases}2x+1 & \text{if } x\leqslant a, \text{ and}\\ \sqrt{x+2} & \text{if } a<x,\end{cases}$$

is continuous on $\mathbb{R}$. Write a complete and well-organized solution.

Question 9. a. Given that $\sqrt{3x^2+4}<f(x)<2-x\sqrt{3}$, for $x<0$, evaluate

$$\lim_{x\to-\infty}\frac{f(x)}{7x+2}.$$

For full marks, write a complete and well-organized solution.

b. Classify the discontinuities of the function g defined by $g(x)=[\![x]\!]-[\![-x]\!]$.

Question 10. Give an example of a function f for which

$$\lim_{x\to 0^+} f(x)=0, \qquad \text{yet} \qquad \lim_{x\to 0^+}\frac{1}{f(x)}$$

is neither ∞ nor $-\infty$.

Solutions to the review exercises

Solution to exercise 1. a. As $x \to 0^-$, $x-1 \to -1^-$ and $x+1 \to 1^-$, so $f(x-1) \to 1$, and $f(x+1) \to 0^+$. Therefore,

$$\lim_{x\to 0} \frac{f(x-1)}{f(x+1)} = \infty.$$

As $x \to 0^+$, $x-1 \to -1^+$ and $x+1 \to 1^+$, so $f(x-1) \to 2$, and $f(x+1) \to 0^-$. Therefore,

$$\lim_{x\to 0} \frac{f(x-1)}{f(x+1)} = -\infty.$$

So the limit in question is undefined.

b. As $x \to 0^\pm$, $1/x \to \pm\infty$, so $g(1/x) \to 2^-$ and hence $1-g(1/x) \to -1^+$. Thus,

$$\lim_{x\to 0} f\Big(1-g(1/x)\Big) = 2.$$

c. Since

$$\lim_{x\to -1^-} \Big\{f(x)g(x)\Big\} = 1\cdot 1 = 1 \qquad \text{and} \qquad \lim_{x\to -1^+} \Big\{f(x)g(x)\Big\} = 2\cdot 1 = 2,$$

the limit in question is undefined.

d. The limits as $x \to 0^\pm$ of $f(x-1)$ are 2 and 1. But as $x \to 0^\pm$, $-x^2 \to 0^-$, and so $g(-x^2) \to 0^+$. Therefore,

$$\lim_{x\to 0^-} \Big\{f(x-1)g(-x^2)\Big\} = 2\cdot 0 = 0 \quad \text{and} \quad \lim_{x\to 0^-} \Big\{f(x-1)g(-x^2)\Big\} = 1\cdot 0 = 0,$$

so the limit in question is equal to zero.

e. As $x \to 0^-$, $f(x)$ has a positive limit (between 1 and 2), and $g(x) \to 0^+$. Therefore,

$$\lim_{x\to 0^-} \frac{f(x)}{g(x)} = \infty.$$

f. As $x \to \infty$, $g(x) \to 2^-$, so

$$\lim_{x\to\infty} (f\circ g)(x) = -\infty.$$

g. As $x \to 0^-$, $g(x) \to 0^+$, so

$$\lim_{x\to 0} (g\circ g)(x) = \infty.$$

As $x \to 0^+$, $g(x) \to \infty$, so

$$\lim_{x\to 0} (g\circ g)(x) = 2.$$

Therefore, the limit in question is undefined.

h. As $x \to -1^-$, $f(x) \to 1$, so $(g\circ f)(x) \to 2$, and thus

$$\lim_{x\to -1^-} \Big\{f(x) + (g\circ f)(x)\Big\} = 1+2 = 3.$$

As $x \to -1^+$, $f(x) \to 2$, so $(g\circ f)(x) \to 1$, and thus

$$\lim_{x\to -1} \Big\{f(x) + (g\circ f)(x)\Big\} = 2+1 = 3.$$

Therefore, the limit in question is equal to 3.

i. As $x \to 2^\pm$, $g(x) \to 1$ and $f(x) \to \pm\infty$, so

$$\lim_{x\to 2} \frac{g(x)}{f(x)} = 0.$$

j. Since (using the factorization of a difference of squares)

$$\begin{aligned}\frac{1}{x+1+\sqrt{3x^2+5x-1}} &= \frac{\sqrt{3x^2+5x-1}-x-1}{3x^2+5x-1-(x+1)^2} \\ &= \frac{\sqrt{3x^2+5x-1}-x-1}{(x+2)(2x-1)},\end{aligned}$$

the limit in question is equal to

$$\left\{\lim_{x\to -2} \frac{f(x)+1}{x+2}\right\}\cdot\left\{\lim_{x\to -2} \frac{\sqrt{3x^2+5x-1}-x-1}{2x-1}\right\} = \left(\tfrac{1}{2}\right)\left(-\tfrac{2}{5}\right) = -\tfrac{1}{5}.$$

The factor $\frac{1}{2}$ is the slope of the tangent line in the figure, and the factor $-\frac{2}{5}$ is obtained by direct substitution.

Solution to exercise 2. a. Since $2\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) - 7 = 6-7 < 0$, and $4x^2-12x+9 = (2x-3)^2 \to 0^+$ as $x \to \frac{3}{2}$, it follows that

$$\lim_{x\to \frac{3}{2}} \frac{2x^2+x-7}{4x^2-12x+9} = -\infty.$$

b. As $\vartheta \to \frac{5}{3}\pi^\pm$, $\sin\vartheta \to -\frac{1}{2}\sqrt{3}$ and $1-2\cos\vartheta \to 0^\mp$. Therefore,

$$\lim_{\vartheta\to\frac{5}{3}\pi^-} \frac{\sin\vartheta}{1-2\cos\vartheta} = -\infty \qquad \text{and} \qquad \lim_{\vartheta\to\frac{5}{3}\pi^+} \frac{\sin\vartheta}{1-2\cos\vartheta} = \infty,$$

so the limit in question is undefined.

c. As $t \to 0^+$, $\sin t \to 0^+$, $t^{-1} \to \infty$ and $\sqrt{1+t^{-2}} \to \infty$, so

$$\lim_{t\to 0^+} \frac{\sin t}{t^{-1}+\sqrt{1+t^{-2}}} = 0.$$

Next, rationalizing the denominator and extracting the negative power of t gives

$$\begin{aligned}\frac{\sin t}{t^{-1}+\sqrt{1+t^{-2}}} &= (\sqrt{1+t^{-2}}-t^{-1})\sin t \\ &= -(\sqrt{t^2+1}+1)\frac{\sin t}{t} \quad \text{if } t<0,\end{aligned}$$

since in that case $\sqrt{t^{-2}} = -t^{-1}$. Therefore,

$$\lim_{t\to 0^+} \frac{\sin t}{t^{-1}+\sqrt{1+t^{-2}}} = -2$$

by arithmetical properties of limits and the fact that $(\sin t)/t \to 1$ as $t \to 0$. The left and right limits are unequal, so the limit in question is undefined.

d. As $t \to 0^-$, $1-e^{-1/t} \to -\infty$, and hence $\pm 1/(1-e^{-1/t}) \to 0$. But if $t<0$ then

$$\frac{1}{1-e^{-1/t}} \leqslant \frac{\cos(1/t)}{1-e^{-1/t}} \leqslant \frac{-1}{1-e^{-1/t}},$$

so plainly

$$\lim_{t\to 0^-} \frac{\cos(1/t)}{1-e^{-1/t}} = 0.$$

As $t \to 0^+$, $1-e^{-1/t} \to 1$, and $\cos(1/t)$ oscillates between -1 and 1, so

$$\lim_{t\to 0^+} \frac{\cos(1/t)}{1-e^{-1/t}}$$

is undefined. Since the left limit is zero and the right limit is undefined, the limit in question is undefined.

e. The numerator and denominator each vanish as $x \to 5$, so the limit cannot be evaluated by direct substitution. Multiplying and dividing the expression in the limit by $4(x-7)(3+\sqrt{2x-1})$, and then factorizing the result, yields

$$\begin{aligned}\lim_{x\to 5} \frac{\dfrac{1}{x-7}+\dfrac{x-3}{4}}{3-\sqrt{2x-1}} &= \lim_{x\to 5} \frac{(4+(x-3)(x-7))(3+\sqrt{2x-1})}{4(x-7)(9-(2x-1))} \\ &= \lim_{x\to 5} \frac{(x-5)^2(3+\sqrt{2x-1})}{-8(x-7)(x-5)} \\ &= 0.\end{aligned}$$

f. The numerator and denominator vanish as $t \to \frac{1}{2}$, so they are divisible by $t-\frac{1}{2}$, or $2t-1$, and factorizing yields

$$\lim_{t\to\frac{1}{2}} \frac{2t^2+t-1}{2t^3+3t^2-1} = \lim_{t\to\frac{1}{2}} \frac{(2t-1)(t+1)}{(2t-1)(t+1)^2} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}.$$

g. Since $1-\cos 3x = 2\sin^2(\frac{3}{2}x)$, it follows that

$$\lim_{x\to 0} \frac{x\sin 2x}{1-\cos 3x} = \frac{4}{9}\cdot\lim_{x\to 0}\left\{\left(\frac{\sin 2x}{2x}\right)\left(\frac{\frac{3}{2}x}{\sin\frac{3}{2}x}\right)^2\right\} = \frac{4}{9}.$$

h. As $x \to 3^-$, $2\pi/x \to \frac{2}{3}\pi^+$, so $\pi\cos(2\pi/x) \to -\frac{1}{2}\pi^-$, and therefore

$$\lim_{x\to 3^-} \tan\left(\pi\cos\left(\frac{2\pi}{x}\right)\right) = \lim_{t\to\frac{1}{2}\pi^-} \tan t = \infty.$$

i. Multiplying and dividing by $(1-3x)^{3/2}+2\sqrt{2}$, and then factorizing and applying arithmetical properties of limits, gives

$$\lim_{x\to-\frac{1}{3}}\frac{(1-3x)^{3/2}-2\sqrt{2}}{9x^2-1}=\lim_{x\to-\frac{1}{3}}\frac{(1-3x)^3-8}{(3x-1)(3x+1)((1-3x)^{3/2}+2\sqrt{2})}$$
$$=\frac{1}{8\sqrt{2}}\lim_{x\to-\frac{1}{3}}\frac{(3x+1)((1-3x)^2+2(1-3x)+4)}{3x+1}$$
$$=\tfrac{3}{4}\sqrt{2}.$$

j. Combining the expressions in the limit and factorizing yields

$$\frac{3}{t^2-t-2}-\frac{t}{t+1}=\frac{3+2t-t^2}{(t-2)(t+1)}=\frac{(3-t)(t+1)}{(t-2)(t+1)}.$$

Therefore,

$$\lim_{t\to-1}\left\{\frac{3}{t^2-t-2}-\frac{t}{t+1}\right\}=\frac{3-(-1)}{-1-2}=-\tfrac{4}{3}.$$

k. If $t=\pi\sqrt{\varphi^{-1}}$, then

$$\lim_{\varphi\to\infty}\varphi\tan^2\!\left(\frac{\pi}{\sqrt{\varphi}}\right)=\pi^2\lim_{t\to0^+}\left(\frac{\sin t}{t}\cdot\frac{1}{\cos t}\right)^2=\pi^2.$$

l. Extracting the dominant powers of r (recall that $\sqrt{r^2}=-r$ since $r<0$) gives

$$\lim_{r\to-\infty}\frac{r+\sqrt{2r^2+2r-3}}{9+3r}=\lim_{r\to-\infty}\frac{1-\sqrt{2+2/r-3/r^2}}{3(3/r+1)}=\tfrac{1}{3}(1-\sqrt{2}).$$

m. Observe that

$$\frac{x^2-1}{|2x^2+x-1|}=\frac{x-1}{|2x-1|}\cdot\frac{x+1}{|x+1|}=\begin{cases}\dfrac{1-x}{|2x-1|} & \text{if } x<-1\text{, and}\\[2mm] \dfrac{x-1}{|2x-1|} & \text{if } -1<x \text{ and } x\neq\frac{1}{2}.\end{cases}$$

Therefore,

$$\lim_{x\to-1^-}\frac{x^2-1}{|2x^2+x-1|}=\tfrac{2}{3}\quad\text{and}\quad\lim_{x\to-1^+}\frac{x^2-1}{|2x^2+x-1|}=-\tfrac{2}{3},$$

which implies that the limit in question is undefined.

n. If $-4<x<-2$, then

$$5x^2-|14x-3|=5x^2+14x-3=(x+3)(5x-1),$$

and

$$2x^3+|15x-9|=2x^3-15x+9=(x+3)(2x^2-6x+3),$$

and hence

$$\lim_{x\to-3}\frac{5x^2-|14x-3|}{2x^3+|15x-9|}=\lim_{x\to-3}\frac{5x-1}{2x^2-6x+3}=-\tfrac{16}{39},$$

o. Multiplying and dividing by $2\sqrt{x+\sqrt{3x}}+\sqrt{4x-1}$ and then extracting the dominant powers of x gives

$$\lim_{x\to\infty}\left\{2\sqrt{x+\sqrt{3x}}-\sqrt{4x-1}\right\}=\lim_{x\to\infty}\frac{4\sqrt{3}+x^{-1/2}}{2\sqrt{1+\sqrt{3x^{-1}}}+\sqrt{4-x^{-1}}}=\sqrt{3}.$$

Solution to exercise 3. Since f is piece-wise rational, it is continuous at all real numbers except possibly -1 and surely not at 1 and 2 (since the latter do not belong to the domain of f). Now $f(-1)=1$,

$$\lim_{x\to-1^-}f(x)=2(-1)+3=1\qquad\text{and}\qquad\lim_{x\to-1^+}f(x)=\frac{2(-1)^2-(-1)+1}{(-1-1)^2}=1,$$

so f is continuous at -1. Next, $2x^2-x+1\to2$ and $(x-1)^2\to0^+$ as $x\to1^-$, so

$$\lim_{x\to1^-}f(x)=\infty,$$

and hence f has an infinite discontinuity at 1. (The limit of $f(x)$ as $x\to1^+$ is equal to $-\frac{1}{6}$, but this is immaterial to the conclusion.) Finally, 2 does not belong to the domain of f, yet

$$\lim_{x\to2}f(x)=\lim_{x\to2}\frac{x-2}{(x-2)(x+5)}=\lim_{x\to2}\frac{1}{x+5}=\tfrac{1}{7},$$

so f has a removable discontinuity at 2. Thus, f has an infinite discontinuity at 1, a removable discontinuity at 2, and is otherwise continuous.

Solution to exercise 4. Since h is defined by a polynomial on $(-\infty,2)$, and by another polynomial on $[2,\infty)$, it follows that h is continuous on $\mathbb{R}$ if, and only if, h is continuous at 2. Now

$$\lim_{x\to2^-}h(x)=a^3+12,\qquad\text{and}\qquad h(2)=\lim_{x\to2^+}h(x)=3a^2+4a,$$

so h is continuous at 2 if, and only if $a^3+12=3a^2+4a$, or $a^3-3a^2-4a+12=0$. Factorizing the left side of this last equation gives

$$a^3-3a^2-4a+12=(a^2-4)(a-3)=(a+2)(a-2)(a-3).$$

Therefore, h is continuous on $\mathbb{R}$ if, and only if, a is -2, 2 or 3.

Solution to exercise 5. The function f is continuous everywhere except possibly b, where

$$\lim_{x\to b^-}f(x)=b^2+6b-18,\quad f(b)=a\quad\text{and}\quad\lim_{x\to b^+}f(x)=5b-6.$$

Hence, f is continuous on $\mathbb{R}$ if, and only if, $b^2+6b-18=5b-6$ and $a=5b-6$. The first equation is equivalent to $b^2+b-12=0$, or $(b+4)(b-3)=0$, whose solutions are -4 (which gives $a=-26$) and 3 (which gives $a=9$). Therefore, f is continuous on $\mathbb{R}$ if, and only if, either $a=-26$ and $b=-4$, or else $a=9$ and $b=3$.

Solution to exercise 6. The given inequality implies that

$$\tfrac{1}{6}-\tfrac{1}{120}x^2<\frac{x-\sin x}{x^3}<\tfrac{1}{6}$$

if $0<x<1$, and therefore also if $-1<x<0$ since each term of the displayed inequality is an even function of x. Since $\lim_{x\to0}\left(\frac{1}{6}-\frac{1}{120}x^2\right)=0$, it follows that

$$\lim_{x\to0}\frac{x-\sin x}{x^3}=\tfrac{1}{6}.$$

Solution to exercise 7. As f is a rational function with a removable discontinuity at 4, the numerator and denominator of $f(x)$ are divisible by $x-4$. Since the graph of f has x-intercepts at $-\frac{5}{3},1$ and vertical asymptotes defined by $x=-5$ and $x=\frac{2}{7}$, the numerator of $f(x)$ is divisible by $(3x+5)(x-1)$ and the denominator of $f(x)$ is divisible by $(x+5)(2x-7)$. Hence, there is a real number α such that

$$f(x)=\frac{\alpha(3x+5)(x-1)(x-4)}{(x+5)(2x-7)(x-4)}.$$

The horizontal asymptote of the graph of f is defined by $y=-4$, so

$$-4=\lim_{x\to\infty}f(x)=\tfrac{3}{2}\alpha,\qquad\text{or}\qquad\alpha=-\tfrac{8}{3},$$

and therefore,

$$f(x)=\frac{8(3x+5)(x-1)(x-4)}{3(x+5)(2x-7)(x-4)}.$$

The y-intercept of the graph of f is given by $f(0)=\frac{8}{21}$. Finally,

$$\lim_{x\to4}f(x)=\frac{8(12+5)(4-1)}{3(4+5)(8-7)}=\tfrac{136}{9}.$$

Solution to exercise 8. a. If k is any integer and

$$k<t<k+1,\qquad\text{then}\qquad -k-1<-t<-k,$$

which implies that

$$[\![t]\!]=k\qquad\text{and}\qquad[\![-t]\!]=-k-1,$$

and therefore

$$f(t)=[\![t]\!]+[\![-t]\!]=k+(-k-1)=-1.$$

It follows that

$$\lim_{t\to x}f(t)=-1$$

for all real numbers x, and that

$$\lim_{t\to x}f(t)=f(x)$$

provided x is not an integer. If k is an integer then $[\![k]\!]=k$ and $[\![-k]\!]=-k$, so $f(k)=[\![k]\!]+[\![-k]\!]=k+(-k)=0$. Hence, the intervals of continuity of f are $(k-1,k)$, where k is an integer, and the graph of f has no (horizontal or vertical) asymptotes.

b. The function f is continuous on $(-\infty, 0)$ and on $(0, \infty)$, and

$$\lim_{x\to 0^-} \frac{3e^x+2}{e^x-1} = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} \frac{3e^x+2}{e^x-1} = \infty,$$

so the vertical asymptote of the graph of f is defined by $x = 0$. Since

$$\lim_{x\to \infty} \frac{3e^x+2}{e^x-1} = 3, \qquad \text{and} \qquad \lim_{x\to -\infty} \frac{3e^x+2}{e^x-1} = -2,$$

the vertical asymptotes of the graph of f are defined by $y = 3$ and $y = -2$.

c. The numerator of $g(x)$ vanishes if $x = -2$, and

$$5x^4 + 20x^3 + 23x^2 + 12x + 12 = (x+2)(5x^3 + 10x^2 + 3x + 6).$$

The second factor on the right also vanishes if $x = -2$, and

$$5x^3 + 10x^2 + 3x + 6 = (x+2)(5x^2+3).$$

Therefore,

$$g(x) = \frac{(x+2)(5x^2+3)}{(3x+2)^3}, \qquad \text{provided} \qquad x \neq -2,$$

which implies that

$$\lim_{x\to -2} g(x) = 0.$$

So g is continuous at every real number besides -2 and $-\frac{2}{3}$, with a removable discontinuity at -2. Since $(x+2)(5x^2+3)$ tends to a positive limit as $x \to -\frac{2}{3}$, it follows that

$$\lim_{x\to -\frac{2}{3}^-} g(x) = -\infty \qquad \text{and} \qquad \lim_{x\to -\frac{2}{3}^+} g(x) = \infty;$$

so the vertical asymptote of the graph of g is defined by $x = -\frac{2}{3}$. Finally,

$$\lim_{x\to\pm\infty} \frac{(x+2)(5x^2+3)}{(3x+2)^3} = \lim_{x\to\pm\infty} \frac{(1+2/x)(5+3/x^2)}{(3+2/x)^3} = \frac{5}{27},$$

so the horizontal asymptote of the graph of g is defined by $y = \frac{5}{27}$.

Solution to exercise 9. a. The statement is true. If ε is any positive number, no matter how small, then $|f(x)| > 1/\varepsilon$ provided x is sufficiently close, but not equal, to a, which implies that $|1/f(x)| < \varepsilon$. Hence, $1/f(x) \to 0$ as $x \to a$.

b. This statement is false because the equation does not imply that a belongs to the domain of f.

c. This statement is true by the definition of continuity.

d. The statement is false; f may oscillate. The statement fails if, for example,

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x > 0, \text{ and} \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

e. The statement is true, since the given equation implies that

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left\{ f(a) + \frac{f(x)-f(a)}{x-a}\cdot(x-a)\right\} = f(a) + \ell\cdot 0 = f(a),$$

which implies that f is continuous at a.

f. The statement is false. For example (taking $f(x) = x$ and $a = -1$)

$$\lim_{x\to -1} x^2 = 1 \qquad \text{but} \qquad \lim_{x\to -1} x = -1 \neq \sqrt{1}.$$

Solution to exercise 10. Let $f(x) = [\![1/x^2]\!]$; *i.e.*, $f(x)$ is the greatest integer which is $\leqslant 1/x^2$. If k is a non-zero integer and

$$\frac{1}{\sqrt{k+1}} < |x| < \frac{1}{\sqrt{k}}, \qquad \text{then} \qquad k < \frac{1}{x^2} < k+1,$$

which implies that

$$f(x) \to k \quad \text{as} \quad |x| \to \frac{1}{\sqrt{k}}^- \qquad \text{and} \qquad f(x) \to k-1 \quad \text{as} \quad |x| \to \frac{1}{\sqrt{k}}^+.$$

It follows that f has a jump discontinuity at $\pm k^{-1/2}$ for every non-zero integer k and $f(x) \to \infty$ as $x \to 0$. Finally, if $|x| > 1$, then $0 < 1/x^2 < 1$, so $f(x) = 0$, which implies that $f(x) \to 0$ as $x \to \pm\infty$.

Solutions to the questions in An old Test 1

Solution to question 1. a. i. As $x \to 1^-$, $x - 1 \to 0^-$, so $f(x) \to -1$ and $f(x-1) \to 0^+$, and hence

$$\lim_{x\to 1^-} \frac{f(x)}{f(x-1)} = -\infty.$$

On the other hand, as $x \to 1^+$, $x - 1 \to -^+$, so $f(x) \to 1$ and $f(x-1) \to 0^-$,

$$\lim_{x\to 1^+} \frac{f(x)}{f(x-1)} = -\infty.$$

Thus, the limit in question is $-\infty$.

ii. As $x \to 0$, $\cos x \to 1^-$, so $2 - \cos x \to 1^+$, and hence

$$\lim_{x\to 0} f(2-\cos x) = 1.$$

iii. The point $(-3, 1)$ is on the graph of f, so $f(-3) = 1$.

iv. As $x \to 0^-$, $f(x) \to 0^+$, so $1/f(x) \to \infty$ and hence

$$\lim_{x\to 0^-} f\big(1/f(x)\big) = 0.$$

v. The tangent line to the graph of f at the point whose x-coordinate is -2 passes through the points $(-4, -6)$ and $(0, 3)$, so its slope is $\frac{9}{4}$. Therefore,

$$\lim_{x\to -2} \frac{f(x) - f(-2)}{x+2} = \frac{9}{4}.$$

b. f is continuous on the intervals $(-\infty, -3)$, $(-3, 1]$ and $(1, \infty)$.

c. f has a removable discontinuity at -3, since

$$f(-3) = 1 \quad\text{and}\quad \lim_{x\to -3} f(x) = -3,$$

and f has a jump discontinuity at 1, since

$$\lim_{x\to 1^-} f(x) = -1 \quad\text{and}\quad \lim_{x\to 1^+} f(x) = 1.$$

Solution to question 2. a. Since

$$\sqrt{5r-1} - \sqrt{2r^2+1} = \frac{-2r^2+5r-2}{\sqrt{5r-1}+\sqrt{2r^2+1}},$$

and $-2r^2 + 5r - 2 = (r-2)(1-2r)$, it follows that

$$\lim_{r\to 2} \frac{\sqrt{5r-1} - \sqrt{2r^2+1}}{r-2} = \lim_{r\to 2} \frac{-2r+1}{\sqrt{5r-1}+\sqrt{2r^2+1}} = -\frac{1}{2},$$

by independence and direct substitution.

b. Since $t^2 - 3t + 2 = (t-1)(t-2)$, and hence

$$\frac{1}{t^2-3t+2} + \frac{1}{t-1} = \frac{t-1}{(t-1)(t-2)},$$

it follows that

$$\lim_{t\to 1} \left\{ \frac{1}{t^2-3t+2} + \frac{1}{t-1} \right\} = \lim_{t\to 1} \frac{1}{t-2} = -1,$$

by independence and direct substitution.

Solution to question 3. a. Since $1 - 4x + 4x^2 = |2x-1|^2$, it follows that

$$\lim_{x\to \frac{1}{2}} \frac{|2x-1|}{1-4x+4x^2} = \lim_{x\to \frac{1}{2}} \frac{1}{|2x-1|} = \infty.$$

by independence, since $|2x-1| \to 0^+$ as $x \to \frac{1}{2}$.

b. As $t \to 0^+$, $\sin t \to 0^+$, $t^{-1} \to \infty$ and $\sqrt{1+t^{-2}} \to \infty$, so

$$\lim_{t\to 0^+} \frac{\sin t}{t^{-1} + \sqrt{1+t^{-2}}} = 0.$$

Next, rationalizing the denominator and extracting a power of t gives

$$\frac{\sin t}{t^{-1} + \sqrt{1+t^{-2}}} = \left(\sqrt{1+t^{-2}} - t^{-1}\right)\sin t = -\left(\sqrt{t^2+1}+1\right)\frac{\sin t}{t}$$

if $t < 0$, since in that case $\sqrt{t^{-2}} = -t^{-1}$. Therefore,

$$\lim_{t\to 0^+} \frac{\sin t}{t^{-1} + \sqrt{1+t^{-2}}} = -2.$$

The left and right limits are unequal, so the limit in question is undefined.

Solution to question 4. a. If $t = 5/x$, then

$$\lim_{x\to -\infty} x \sin\left(\frac{5}{x}\right) = 5 \lim_{t\to 0^-} \frac{\sin t}{t} = 5.$$

b. If $a = \sqrt{3x^2-2x+7}$ and $b = \sqrt{3x^2-5x+9}$, then $a^2 - b^2 = 3x - 2$; thus, extracting dominant powers gives

$$\lim_{x\to\infty}(a-b) = \lim_{x\to\infty} \frac{a^2-b^2}{a+b} = \lim_{x\to\infty} \frac{3-2/x}{\sqrt{3-2/x+7/x^2}+\sqrt{3-5/x+9/x^2}}$$
$$= \tfrac{1}{2}\sqrt{3}.$$

c. Writing $2x$ as $\log(e^{2x})$, combining the logarithms, and then multiplying and dividing by e^{-2x}, gives

$$\lim_{x\to\infty} \log\frac{e^{2x}}{e^{2x-3}+1} = \lim_{x\to\infty} \log\frac{1}{e^{-3}+e^{-2x}} = \log\frac{1}{e^{-3}} = 3.$$

d. If $a = \sqrt[3]{5x^3-3x^2+1}$ and $b = x\sqrt[3]{5}$ then $a^3 - b^3 = -3x^2 + 1$; thus, extracting dominant powers gives

$$\lim_{x\to\infty}(a-b) = \lim_{x\to\infty} \frac{a^3-b^3}{a^2+ab+b^2}$$
$$= \lim_{x\to\infty} \frac{-3+1/x^2}{(5-3/x+1/x^3)^{2/3} + 5^{1/3}(5-3/x+1/x^3)^{1/3} + 5^{2/3}}$$
$$= -\tfrac{1}{5}\sqrt[3]{5}.$$

Solution to question 5. a. The numerator and denominator each vanish if $x = -2$, and factorizing gives

$$\lim_{x\to -2} \frac{3x^3+7x^2-4}{3x^2+5x-2} = \lim_{x\to -2} \frac{(x+2)(3x^2+x-2)}{(x+2)(3x-1)} = -\frac{8}{7},$$

by independence and direct substitution.

b. Applying the identity $1 - \cos(x) = 2\sin^2(\frac{1}{2}x)$, and then multiplying and dividing by $\sin^4(\frac{1}{2}\pi\vartheta)$, $(\frac{1}{2}\pi)^4$ and $(\frac{3}{2}\vartheta)^2$, gives

$$\frac{1-\cos(1-\cos\pi\vartheta)}{\vartheta^2(1-\cos 3\vartheta)} = \left(\frac{\sin(\sin^2(\frac{1}{2}\pi\vartheta))}{\sin^2(\frac{1}{2}\pi\vartheta)}\right)^2 \cdot \left(\frac{\sin(\frac{1}{2}\pi\vartheta)}{\frac{1}{2}\pi\vartheta}\right)^4 \cdot \left(\frac{\frac{3}{2}\vartheta}{\sin(\frac{3}{2}\vartheta)}\right)^2 \cdot \frac{1}{36}\pi^4.$$

As $\vartheta \to 0$, the limit of each of the first three factors on the right side is 1, so

$$\lim_{\vartheta\to 0} \frac{1-\cos(1-\cos\pi\vartheta)}{\vartheta^2(1-\cos 3\vartheta)} = \frac{1}{36}\pi^4.$$

c. If $n = 1$, the expression in the limit is identically zero (if $x \neq 1$), so the limit is zero. If $n \geqslant 2$, then

$$x^n - nx + n - 1 = x^n - 1 - n(x-1) = (x-1)(x^{n-1} + x^{n-2} + \cdots + 1 - n).$$

The last factor on the right vanishes is $x = 1$, and factorizing gives

$$x^{n-1} + x^{n-2} + \cdots + 1 - n = (x-1)(x^{n-2} + 2x^{n-3} + \cdots + n - 1)$$

(the factor on the right has one term if $n = 2$, two terms if $n = 3$, and so on). Therefore,

$$\lim_{x\to 1} \frac{x^n - nx + n - 1}{(x-1)^2} = \lim_{x\to 1}(x^{n-2} + 2x^{n-3} + \cdots + n - 1)$$
$$= 1 + 2 + \cdots + n - 1.$$

Since $2(1+2+\cdots+n-1) = n(n-1)$, as is seen by adding the first and last terms, then the second and second last terms, *&c.*, the limit is equal to $\frac{1}{2}n(n-1)$.

d. If $a = t\sqrt{x^2+4}$ and $b = x\sqrt{t^2+4}$ then $a^2 - b^2 = 4(t^2 - x^2) = 4(t-x)(t+x)$, so that

$$\frac{t}{\sqrt{t^2+4}} - \frac{x}{\sqrt{x^2+4}} = \frac{t\sqrt{x^2+4} - x\sqrt{t^2+4}}{\sqrt{(t^2+4)(x^2+4)}} = \frac{4(t-x)(t+x)}{(a+b)\sqrt{(t^2+4)(x^2+4)}}.$$

As $t \to x$, $a + b \to 2x\sqrt{x^2+4}$, and so

$$\lim_{t\to x} \frac{\dfrac{t}{\sqrt{t^2+4}} - \dfrac{x}{\sqrt{x^2+4}}}{t-x} = \lim_{t\to x} \frac{4(t+x)}{(a+b)\sqrt{(t^2+4)(x^2+4)}} = \frac{4}{(x^2+4)^{3/2}}.$$

Solution to question 6. a. Since $3x^2-x+1=\frac{1}{12}(6x-1)^2+\frac{11}{12}$ is positive for all real values of x, the domain of f is $\mathbb{R}\setminus\{\frac{3}{2}\}$. As $x\to\frac{3}{2}^{\pm}$ $x+\sqrt{3x^2-x+1}\to 4$ and $3x-2\to 0^{\pm}$, so so

$$\lim_{x\to\frac{3}{2}^-}\frac{x+\sqrt{3x^2-x+1}}{2x-3}=-\infty \qquad\text{and}\qquad \lim_{x\to\frac{3}{2}^+}\frac{x+\sqrt{3x^2-x+1}}{2x-3}=\infty.$$

Therefore the only vertical asymptote of the graph of f is defined by $x=\frac{3}{2}$. Next, since $\sqrt{x^2}=\pm x$ for $x\gtrless 0$,

$$\lim_{x\to\pm\infty}\frac{x+\sqrt{3x^2-x+1}}{2x-3}\lim_{x\to\pm\infty}\frac{1\pm\sqrt{3-1/x+1/x^2}}{2-3/x}=\tfrac{1}{2}(1\pm\sqrt{3}),$$

so the horizontal asymptotes of the graph of f are defined by $y=\frac{1}{2}(1\pm\sqrt{3})$.

b. The function g is continuous except at integer multiples of π (where $x\sin x=0$). First consider odd multiples of π. As $x\to(2k+1)\pi^{\pm}$, $1-\cos x\to 2$ and $\sin x\to 0\mp$, and since $(2k+1)\pi$ has the same sign as k,

$$\lim_{x\to(2k+1)\pi^{\pm}}\frac{1-\cos x}{x\sin x}=\begin{cases}\pm\infty & \text{if } k<0\text{, and}\\ \mp\infty & \text{if } k\geqslant 0.\end{cases}$$

So the graph of g has vertical asymptotes defined by $x=(2k+1)\pi$, where k is an integer. It follows that

$$\lim_{x\to\pm\infty}g(x)$$

is undefined, so the graph of g has no horizontal asymptotes. At the origin, the identity $1-\cos x=2\sin^2(\frac{1}{2}x)$ gives

$$\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\frac{1}{2}\lim_{x\to 0}\left\{\left(\frac{\sin(\frac{1}{2}x)}{\frac{1}{2}x}\right)^2\cdot\frac{x}{\sin x}\right\}=\frac{1}{2},$$

so g has a removable discontinuity at 0. If $k\neq 0$ is an integer then, again using the identity $1-\cos t=2\sin^2(\frac{1}{2}t)$,

$$\lim_{x\to 2\pi k}\frac{1-\cos x}{x\sin x}=\lim_{t\to 0}\left\{\frac{\sin(\frac{1}{2}t)}{\frac{1}{2}t}\cdot\frac{t}{\sin t}\cdot\frac{\sin(\frac{1}{2}t)}{t+2\pi k}\right\}=0,$$

where $t=x-2\pi k$, so g has a removable discontinuity at $2k\pi$. Therefore, the only asymptotes of the graph of g are the vertical asymptotes defined by $x=(2k+1)\pi$, where k is an integer.

c. By inspection, -1 is a common zero of the numerator and denominator of $h(x)$, and factorizing gives

$$h(x)=\frac{2x^3+3x^2-1}{4x^3-3x+1}=\frac{(x+1)^2(2x-1)}{(x+1)(2x-1)^2}=\frac{x+1}{2x-1},\qquad\text{if } x\neq -1.$$

Then

$$\lim_{x\to\frac{1}{2}^{\pm}}h(x)=\pm\infty,\qquad \lim_{x\to -1}h(x)=0\qquad\text{and}\qquad \lim_{x\to\pm\infty}h(x)=\tfrac{1}{2}$$

so the asymptotes of the graph of g are defined by $y=\frac{1}{2}$ (horizontal) and $x=\frac{1}{2}$ (vertical), the discontinuity at -1 being removable.

d. The function h is defined at all positive real numbers besides $\frac{5}{2}$, where

$$\lim_{x\to\frac{5}{2}^{\pm}}h(x)=\mp\infty$$

(since $3/\sqrt{\frac{5}{2}}=3\sqrt{\frac{2}{5}}=\frac{3}{5}\sqrt{10}$ lies between $\frac{9}{5}$ and $\frac{12}{5}$, it certainly lies between $\frac{1}{2}\pi$ and π, so the numerator of $h(x)$ is negative near $\frac{5}{2}$). Since $-x^2\leqslant\cos\left(3x^{-1/2}\right)\leqslant x^2$ if $x>0$, it is plain that $h(x)\to 0$ as $x\to 0^+$. Thus, the only vertical asymptote of the graph of f is defined by $x=\frac{5}{2}$. If $t=3x^{-1/2}$ then $x=9/t^2$, and so

$$\begin{aligned}h(x)-\tfrac{1}{2}x&=\frac{x^2\cos\left(\frac{3}{\sqrt{x}}\right)}{2x-5}-\tfrac{1}{2}x=\frac{81\cos t}{t^4(18/t^2-5)}-\frac{9}{2t^2}\\&=\frac{9(\cos t-1)}{2t^2}+\frac{5}{4\left(1-\frac{5}{18}t^2\right)}=-\frac{9}{4}\left(\frac{\sin(\frac{1}{2}t)}{\frac{1}{2}t}\right)^2+\frac{5}{4\left(1-\frac{5}{18}t^2\right)}.\end{aligned}$$

As $x\to\infty$, $t\to 0^+$, so

$$\lim_{x\to\infty}\left\{h(x)-\tfrac{1}{2}x\right\}=-\tfrac{9}{4}+\tfrac{5}{4}=-1,$$

which implies that the line defined by $y=\frac{1}{2}x-1$ is the oblique asymptote of the graph of h.

Solution to question 7. The function f is continuous except possibly at -2, 1 and 3. Since

$$\lim_{x\to-2^{\pm}}f(x)=\lim_{x\to-2^{\pm}}\frac{x}{x+2}=\mp\infty,$$

it follows that f has an infinite discontinuity at -2. Next,

$$\lim_{x\to 1^-}f(x)=\frac{1}{1+2}=\tfrac{1}{3}\qquad\text{and}\qquad\lim_{x\to 1^+}f(x)=\cos(\pi)=-1,$$

so f has a jump discontinuity at 1. Finally,

$$\lim_{x\to 3^-}f(x)=\cos(\tfrac{1}{3}\pi)=\tfrac{1}{2}\qquad\text{and}\qquad f(3)=\lim_{x\to 3^+}f(x)=\frac{1}{3-1}=\tfrac{1}{2},$$

so f is continuous at 3. Therefore, f has an infinite discontinuity at -2 and a jump discontinuity at 1; otherwise, f is continuous.

Solution to question 8. The function f is continuous at all real numbers, except possibly at a, provided $a\geqslant -2$ (otherwise, f is ill-defined). Now

$$f(a)=\lim_{x\to a^-}f(x)=2a+1,\qquad\text{and}\qquad\lim_{x\to a^+}f(x)=\sqrt{a+2},$$

so f is continuous at a if, and only if, $\sqrt{a+2}=2a+1$. Equivalently, $a\geqslant-\frac{1}{2}$ and

$$a+2=(2a+1)^2,\quad\textit{i.e.},\quad a+2=4a^2+4a+1,\quad\text{or}\quad 4a^2+3a-1=0.$$

As $4a^2+3a-1=(a+1)(4a-1)$, f is continuous on $\mathbb{R}$ if, and only if, $a=\frac{1}{4}$.

Solution to question 9. a. If $x<-\frac{2}{7}$, then $\sqrt{3x^2+4}<f(x)<2-x\sqrt{3}$ and $7x+2<0$, so

$$\frac{2-x\sqrt{3}}{7x+2}<\frac{f(x)}{7x+2}<\frac{\sqrt{3x^2+4}}{7x+2},\qquad\text{or}\qquad\frac{2/x-\sqrt{3}}{7+2/x}<\frac{f(x)}{7x+2}<-\frac{\sqrt{3+4/x^2}}{7+2/x},$$

the latter since $\sqrt{x^2}=-x$ as $x<0$. Now

$$\lim_{x\to-\infty}\frac{2/x-\sqrt{3}}{7+2/x}=-\tfrac{1}{7}\sqrt{3}\qquad\text{and}\qquad\lim_{x\to-\infty}\frac{-\sqrt{3+4/x^2}}{7+2/x}=-\tfrac{1}{7}\sqrt{3},$$

so plainly

$$\lim_{x\to-\infty}\frac{f(x)}{7x+2}=-\tfrac{1}{7}\sqrt{3}.$$

b. If k is any integer and $k<t<k+1$ then $-k-1<-t<-k$, and hence $g(t)=[\![t]\!]+[\![-t]\!]=k+(-k-1)=-1$; on the other hand $g(k)=[\![k]\!]+[\![-k]\!]=k+(-k)=0$. Therefore,

$$\lim_{t\to x}g(t)=-1$$

for all real numbers x, and it follows the discontinuities of g are the integers, and each of these discontinuities is removable.

Solution to question 10. If $f(x)=x\cos(\pi/x)$, then $-x\leqslant f(x)\leqslant x$ if $x>0$, so plainly $f(x)\to 0$ as $x\to 0^+$. However, if k is an even positive integer then

$$\frac{1}{f(1/k)}=k,\qquad\text{so}\qquad\lim_{x\to 0^+}\frac{1}{f(x)}\qquad\text{is not}\qquad -\infty,$$

and if k is an odd positive integer then

$$\frac{1}{f(1/k)}=-k,\qquad\text{so}\qquad\lim_{x\to 0^+}\frac{1}{f(x)}\qquad\text{is not}\qquad \infty.$$