A sample test

Question 1. — Let $f(x) = x\sqrt{x+1}$.

a. Use the definition of the derivative to compute $f'(x)$.

b. Write an equation of the line tangent to the graph of f at the point with x coordinate 2.

Question 2. — a. Let f be the function defined by

$$f(x) = \begin{cases} x^p \cos(\pi/x) & \text{if } x > 0, \text{ and} \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

Find all values of p for which f is differentiable at 0, but f' is not continuous at 0. For full credit, write a complete and well-organized solution.

b. Using basic properites of the logarithm, show that if a is a real number and b, c are positive real numbers, then

$$\lim_{x\to\infty} \frac{(\log x)^a}{x^b} = 0, \qquad \lim_{y\to 0^+} \left\{ y^b (-\log y)^2 \right\} = 0 \qquad \text{and} \qquad \lim_{z\to\infty} \frac{z^a}{e^{bz^c}} = 0.$$

Question 3. — Show that if f and g are rational functions whose graphs are tangent where $x = a$, then $f(x) - g(x) = (x-a)^2 h(x)$, where h is a rational function whose domain includes a.

Question 4. — Compute each indicated derivative. For full credit, follow all instructions and use the variable and function names as given.

a. Find $\dfrac{dy}{dx}$, where $y = \sqrt[x]{x+1} - \dfrac{x(x^3-5)^3(5x^6+1)^2}{(4x^7+20)^2}$.

b. Find $\dfrac{dx}{dy}$, where $x^y = y^x$.

c. Given $f(x) = \cot\left(2x + \log_7(3x + \sec 5x)\right)$, find $f'(x)$.

d. Find $\dfrac{dw}{dr}$, if $\dfrac{\sqrt{w}}{\tan(r)+1} = \ln(rw)$.

e. Find and simplify $\left.\dfrac{dz}{dy}\right|_{y=\frac{1}{6}\pi}$, where $z = \log_{(\sin y)}\left(e^{\cos 3y} + \tan(\tfrac{3}{2}y)\right)$.

Question 5. — Let $\mathscr{C}$ be the curve defined by $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$.

a. Find $\dfrac{dy}{dx}$ and simplify your answer as much as possible.

b. Find all points on $\mathscr{C}$ where $y = 2$. Write an equation of the tangent line to $\mathscr{C}$ at each point.

c. Find all points on $\mathscr{C}$ at which the tangent line has slope 3.

d. There are two tangent lines to $\mathscr{C}$ which pass through the point $(-2,-2)$. Write an equation of each of these tangent lines.

Question 6. — The tangent line to the graph of a function f at the point where $x = 1$ is defined by $2x + 3y = 5$. Also, $f''(1) = 7$ and $f'''(1) = -9$. Let $g(x) = f'\left(xf(x)\right)$ and $h(x) = f\left(e^{2x}\right)$.

a. Find an equation of the line tangent to the graph of g where $x = 1$.

b. Find an equation of the line tangent to the graph of h where $x = 0$.

c. Find $g''(1)$.

Question 7. — For each equation, find all points on its graph where the tangent line is horizontal.

a. $y = \sin(2x) - 2\sin x$

b. $y = x^{x^2}$

Question 8. — A circle has centre $(0,0)$ and radius $f > 0$. A parabola has vertex $(0,0)$ and focus $(0,f)$. Find all lines tangent to both curves. [This is due to Rubén Calzadilla-Badra.]

Question 9. — Determine a formula defining $f^{(n)}(x)$, where $f(x) = x^2 e^{-3x}$.

Question 10. — a. State and prove the product rule for differentiation.

b. State and prove the power rule for negative rational exponents.

Another sample test

Question 1. — Given $f(x) = \dfrac{x}{2x+1}$.

a. Use the definition of the derivative to find $f'(x)$.

b. Find all points on the graph of f at which the tangent line has y intercept 2.

Question 2. — a. Use the definition of the logarithm in terms of area to show that

$$\frac{d}{dx}\left\{\log(x)\right\} = \frac{1}{x} \qquad \text{if} \quad x > 0.$$

b. Using the basic properties of the logarithm and exponents, derive the formula

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

Question 3. — Show that if the functions f, g are differentiable at a, $f(a) = g(a)$, and $f(x) < g(x)$ for $x \neq a$, then $f'(a) = g'(a)$.

Question 4. — Find the indicated derivative. Use the variable names and notation as given.

a. Find $\dfrac{du}{dr}$ if $u = \left(\sqrt[3]{r^5} - \dfrac{4}{\sqrt[5]{r^3}} + e^\pi\right)\sin(\pi^e)$.

b. Find $\dfrac{dp}{dq}$ if $p = \tan\left(q^3\sin(q) + \sqrt{\cos(\ln q)}\right)$.

c. Find $f'(x)$ if $f(x) = \dfrac{\sec x + \log_2(\cos x)}{\sqrt{x^3 - x^2 + 7}}$.

d. Find $\dfrac{dx}{dy}$ if $y\sin(x^2y) = 1 - \cos(2x + 3y)$.

e. Find $\dfrac{dy}{dx}$ if $y = \tan\left((\sin x)^{\cot x}\right)$.

Question 5. — Let $\mathscr{H}$ be the curve defined by $(x^2+y^2)^2 - 12(x^2+y^2) = 36x^2$.

a. Find and simplify $\dfrac{dy}{dx}$.

b. Find equations of all tangents to $\mathscr{H}$ where it meets the line defined by $y = x$.

c. Find all points at which the line tangent to $\mathscr{H}$ is horizontal.

d. Find all points at which the line tangent to $\mathscr{H}$ is vertical.

Question 6. — Find all real numbers α, β and γ such that if

$$y = \tan^2(\gamma x), \qquad \text{then} \qquad \frac{d^2y}{dx^2} = 8(1+\alpha y)(1+\beta y).$$

Question 7. — Find all values of x at which the tangent to the graph of f is horizontal.

a. $f(x) = x(\log x)^2$

b. $f(x) = (x-2)^2(x^2+2x-12)^3$

Question 8. — The position of a particle along the x axis is described by the equation

$$x = (2t-1)^3(t-2)^6, \qquad \text{for} \qquad t \geqslant 0,$$

where t is measured in seconds and x is measured in metres.

a. When is the particle at rest?

b. When is the particle moving to the left?

c. What distance does the particle travel during the first three seconds?

Question 9. — Find the monic poynomial of degree four, $y = a + bx + cx^2 + dx^3 + x^4$, whose graph is tangent to the line $y = -3x - 2$ where $x = -1$, and is tangent to the line $y = 6 - 7x$ where $x = 1$.

Question 10. — a. State and prove the chain rule for differentiation.

b. Prove that if $y = \sin(\vartheta)$ then $\frac{dy}{dx} = \cos(\vartheta)$.

Solutions to 'A sample test'

Solution to question 1. — a. If $f(x) = x\sqrt{x+1}$, then

$$f(t) - f(x) = t\sqrt{t+1} - x\sqrt{x+1} = \frac{t^2(t+1) - x^2(x+1)}{t\sqrt{t+1} + x\sqrt{x+1}},$$

where the numerator is $t^3 - x^3 + t^2 - x^2 = (t-x)(t^2 + tx + x^2 + t + x)$. Therefore,

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x} \frac{t^2 + tx + x^2 + t + x}{t\sqrt{t+1} + x\sqrt{x+1}} = \frac{3x^2 + 2x}{2x\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}.$$

b. Since $f(2) = 2\sqrt{3}$ and $f'(2) = \frac{8}{2\sqrt{3}} = \frac{4}{3}\sqrt{3}$, it follows that the tangent line is defined by

$$y - 2\sqrt{3} = \tfrac{4}{3}\sqrt{3}(x-2), \qquad \text{or equivalently} \qquad 4x - \sqrt{3}\,y = 2.$$

Solution to question 2. — a. The function f is differentiable at 0 if, and only if,

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 0, \qquad \text{or equivalently} \qquad \lim_{x\to 0^+} x^{p-1}\cos(\pi/x) = 0.$$

This is impossible if $p \leqslant 1$, since then $f(1/n)/(1/n) = (-1)^n n^{1-p}$, which is alternately positive and negative and has absolute value $\geqslant 1$ if n is a positive integer. On the other hand, if $p > 1$ then

$$-x^{p-1} \leqslant x^{p-1}\cos(\pi/x) \leqslant x^{p-1} \quad \text{if} \quad x > 0, \qquad \text{and hence} \qquad \lim_{x\to 0^+} x^{p-1}\cos(\pi/x) = 0,$$

since $\pm x^{p-1} \to 0$ as $x \to 0^+$. So f is differentiable at 0 if, and only if, $p > 1$, in which case $f'(0) = 0$.

Next, if $x < 0$ then $f'(x) = 0$, and if $x > 0$ then

$$f'(x) = px^{p-1}\cos(\pi/x) + \pi x^{p-2}\sin(\pi/x).$$

By the same reasoning as above, $f'(x) \to 0$ as $x \to 0^+$ if, and only if, $p > 2$.

Therefore, f is differentiable at 0 but f' is not continuous at 0 if, and only if, $1 < p \leqslant 2$.

b. If $x > 1$ then the region below the graph of $y = 1/x$ and above the interval $[1, x]$, whose area is plainly positive and is equal to $\log(x)$, is contained in a rectangle of base $x - 1$ and height 1. Moreover, this rectangle contains a rectangle of base $\frac{1}{2}(x-1)$, height $1 - 2/(x+1) = (x-1)/(x+1)$, and area $\frac{1}{2}(x-1)^2/(x+1)$, which lies outside the region, except for its bottom left corner. Therefore, $0 < \log(x) < x - 1 < x$. Replacing x by $x^{b/(2a)}$ gives $0 < \log\left(x^{b/(2a)}\right) < x^{b/(2a)}$, or

$$0 < \frac{b}{2a}\log(x) < x^{b/(2a)}, \qquad \text{and hence} \qquad 0 < \frac{(\log x)^a}{x^b} < \left(\frac{2a}{b}\right)^a x^{-b/2},$$

where the last inequality is obtained by multiplying by $2a/b$, raising to the power a and then dividing by x^b. This is assuming $a, b > 0$. If $a \leqslant 0$ and $x \geqslant e$ then $0 < (\log x)^a x^{-b} \leqslant x^{-b}$, and if $b > 0$ then $x^{-b/2} \to 0$ and $x^{-b} \to 0$ as $x \to \infty$, so regardless of the value of a, it is plain that

$$\lim_{x\to\infty} \frac{(\log x)^a}{x^b} = 0.$$

If $y = 1/x$ then $y \to 0^+$ as $x \to \infty$, $\log(x) = -\log(y)$ and $x^{-b} = y^b$, so in terms of y this last limit is

$$\lim_{y\to 0^+} \left\{ y^b(-\log y)^a \right\} = 0.$$

If c is a positive real number then, replacing a by a/c in the first limit gives

$$\lim_{x\to\infty} \frac{(\log x)^{a/c}}{x^b} = 0.$$

Let $z = (\log x)^{1/c}$, so that $z \to \infty$ as $x \to \infty$, $(\log x)^{a/c} = z^a$ and $x^b = e^{bz^c}$; then the preceding limit, expressed in terms of z, is

$$\lim_{z\to\infty} \frac{z^a}{e^{bz^c}} = 0.$$

Solution to question 3. — First note that $f(x) - g(x) = N(x)/D(x)$ for polynomials $N(x)$ and $D(x)$ with $D(a) \neq 0$. Since the graphs of f and g are tangent where $x = a$, $f(a) = g(a)$ and $f'(a) = g'(a)$. Then $N(a) = 0$, so division gives $N(x) = (x-a)\big((x-a)q(x) + r\big)$, where $q(x)$ is a polynomial and r is a real number. As $f(a) - g(a) = 0$ and $f'(a) - g'(a) = 0$, the definition of the derivative implies that

$$0 = \lim_{x\to a} \frac{f(x) - g(x)}{x - a} = \lim_{x\to a} \frac{N(x)}{(x-a)D(x)} = \lim_{x\to a} \frac{(x-a)q(x) + r}{D(x)} = \frac{r}{D(a)}.$$

So $r = 0$, and therefore, $f(x) - g(x) = (x-a)^2 q(x)/D(x)$, as required.

Solution to question 4. — a. Write $y = u - w$, so that logarithmic differentiation gives

$$\frac{du}{dx} = \sqrt[x]{x^2+1}\,\frac{d}{dx}\left\{\frac{\log(x^2+1)}{x}\right\} = \sqrt[x]{x^2+1}\left\{\frac{2x}{x(x^2+1)} - \frac{\log(x^2+1)}{x^2}\right\}$$
$$= x^{-2}(x^2+1)^{1/x-1}\left\{2x^2 - (x^2+1)\log(x^2+1)\right\},$$

$$\frac{dw}{dx} = w\frac{d}{dx}\left\{\log|x| + 3\log|x^3-5| + 2\log|5x^6+1| - 2\log|4x^7+20|\right\}$$
$$= \frac{x(x^3-5)^3(5x^6+1)^2}{(4x^7+20)^2}\left\{\frac{1}{x} + \frac{9x^2}{x^3-5} + \frac{60x^5}{5x^6+1} - \frac{56x^6}{4x^7+20}\right\},$$

and thus

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dw}{dx}.$$

b. Writing the equation as $y\log x - x\log y = 0$ and then differentiating implicitly with respect to y gives

$$\frac{dx}{dy} = -\frac{\log x - x/y}{y/x - \log y} = \frac{x(x - y\log x)}{y(y - x\log y)}.$$

c. The chain rule gives

$$f'(x) = -\csc^2\left(2x + \log_7(3x + \sec 5x)\right)\left\{2 + \frac{3 + 5\sec 5x\tan 5x}{(3x + \sec 5x)\log(7)}\right\}.$$

d. Writing the equation as $(1 + \tan r)\log(rw) - \sqrt{w} = 0$ and differentiating implicitly with respect to r gives

$$\frac{dw}{dr} = -\frac{\log(rw)\sec^2 r + (1 + \tan r)/r}{(1 + \tan r)/w - \frac{1}{2}w^{-1/2}} = \frac{2w\left(r\log(rw)\sec^2 r + \tan r + 1\right)}{r\left(\sqrt{w} - 2(1 + \tan r)\right)}.$$

e. If

$$z = \log_{(\sin y)}\left(e^{\cos 3y} + \tan(\tfrac{3}{2}y)\right) = \frac{\log\left(e^{\cos 3y} + \tan(\frac{3}{2}y)\right)}{\log(\sin y)},$$

then

$$\frac{dz}{dy} = \frac{-3e^{\cos 3y}\sin 3y + \frac{3}{2}\sec^2(\frac{3}{2}y)}{\left(e^{\cos 3y} + \tan(\frac{3}{2}y)\right)\log(\sin y)} - \frac{\cos(y)\log\left(e^{\cos 3y} + \tan(\frac{3}{2}y)\right)}{\sin(y)\left(\log(\sin y)\right)^2},$$

and so

$$\left.\frac{dz}{dy}\right|_{y=\frac{1}{6}\pi} = \frac{-3+3}{(1+1)\log(\frac{1}{2})} - \frac{\frac{1}{2}\sqrt{3}\log(1+1)}{\frac{1}{2}\left(\log(\frac{1}{2})\right)^2} = -\frac{\sqrt{3}}{\log 2}.$$

Solution to question 5. — a. Implicit differentiation gives

$$\frac{dy}{dx} = -\frac{2x - 2y - 2}{2y - 2x - 2} = \frac{x - y - 1}{x - y + 1}.$$

b. If $y = 2$ on $\mathscr{C}$, then $x^2 - 6x + 1 = 0$, or $(x-3)^2 = 8$, so $x = 3 \pm 2\sqrt{2}$, in which case $x - y = 1 \pm 2\sqrt{2}$ and the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{\substack{x=3\pm2\sqrt{2}\\ y=2}} = \frac{\pm 2\sqrt{2}}{2 \pm 2\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \pm 1} = 2 \mp \sqrt{2}.$$

The tangent line to $\mathscr{C}$ at $(3 + 2\sqrt{2}, 2)$ is defined by

$$y = 2 + (2 - \sqrt{2})(x - 3 - 2\sqrt{2}), \qquad \text{or} \qquad (2 - \sqrt{2})x - y = -4 + \sqrt{2},$$

and the tangent line to $\mathscr{C}$ at $(3 - 2\sqrt{2}, 2)$ is defined by

$$y = 2 + (2 + \sqrt{2})(x - 3 + 2\sqrt{2}), \qquad \text{or} \qquad (2 + \sqrt{2})x - y = -4 - \sqrt{2}.$$

c. The tangent line to $\mathscr{C}$ has slope 3 where $x - y - 1 = 3(x - y) + 3$, or $x - y = -2$. The equation of $\mathscr{C}$ is equivalent to $4y = (x - y - 1)^2$, so $x - y = -2$ on the curve where $4y = 9$, which gives $y = \frac{9}{4}$ and $x = \frac{1}{4}$. Therefore, the tangent line to $\mathscr{C}$ has slope 3 at the point $\left(\frac{1}{4}, \frac{9}{4}\right)$, and at no other point.

d. The tangent line to $\mathscr{C}$ at (x, y) passes through the point $(-2, -2)$ if, and only if,

$$\frac{x - y - 1}{x - y + 1} = \frac{y + 2}{x + 2}, \qquad \text{or equivalently} \qquad (x - y)^2 = x + y + 4,$$

and $(x - y)^2 = 2(x + y) - 1$ (where the latter is equivalent to the equation defining $\mathscr{C}$). Combining these equations gives $x + y = 5$, and hence $(x - y)^2 = 9$, or $x - y = \pm 3$. The pair of equations $x + y = 5$, $x - y = 3$ gives $x = 4$, $y = 1$ and $\frac{dy}{dx} = \frac{1}{2}$, the tangent line to $\mathscr{C}$ at $4, 1)$ is defined by $x - 2y = 2$. The pair of equations $x + y = 5$, $x - y = -3$ gives $x = 1$, $y = 4$ and $\frac{dy}{dx} = 2$, the tangent line to $\mathscr{C}$ at $(1, 4)$ is defined by $2x - y = -2$.

Solution to question 6. — From the given equation of the tangent line to the graph of f where $x = 1$, and the other given information, it follows that

$$f(1) = 1, \qquad f'(1) = -\tfrac{2}{3}, \qquad f''(1) = 7 \qquad \text{and} \qquad f'''(1) = -9.$$

a. Since $g(x) = f'(xf(x))$,

$$g(1) = f'(f(1)) = f'(1) = -\tfrac{2}{3}, \qquad \text{and} \qquad g'(x) = f''(xf(x))\big(f(x) + xf'(x)\big),$$

so

$$g'(1) = f''(f(1))\big(f(1) + f'(1)\big) = f''(1)\big(1 - \tfrac{2}{3}\big) = \tfrac{7}{3}.$$

Therefore, $7x - 3y = 9$ is an equation of the line tangent to the graph of g at the point where $x = 1$.

b. Since $h(x) = f\left(e^{2x}\right)$,

$$h(0) = f(e^{2\cdot 0}) = f(1) = 1, \qquad \text{and} \qquad h'(x) = f'(e^{2x}) \cdot e^{2x} \cdot 2, \qquad \text{so} \qquad h'(1) = f'(1) \cdot 2 = -\tfrac{4}{3}.$$

Therefore, $4x + 3y = 3$ is an equation of the tangent line to the graph of h at the point where $x = 0$.

c. Since $g'(x) = f''(xf(x))(f(x) + xf'(x))$, it follows that

$$g''(x) = f'''(xf(x))\big(f(x) + xf'(x)\big)^2 + f''(xf(x))\big(2f'(x) + xf''(x)\big),$$

and so

$$g''(1) = f'''(1)\big(1 - \tfrac{2}{3}\big)^2 + f''(1)\big(-\tfrac{4}{3} + 7\big) = -9 \cdot \tfrac{1}{9} + 7 \cdot \tfrac{17}{3} = \tfrac{116}{3}.$$

Solution to question 7. — a. If $y = \sin(2x) - 2\sin x$, then

$$\frac{dy}{dx} = 2\cos(2x) - 2\cos x = 2(2\cos^2 x - \cos x - 1) = 2(2\cos x + 1)(\cos x - 1),$$

which is equal to zero if, and only if, either $\cos x = -\frac{1}{2}$, *i.e.*, x is $\pm\frac{2}{3}\pi + 2k\pi$, or else $\cos x = 1$, *i.e.*, x is $2k\pi$, where in each case k is an integer. If $x = \frac{2}{3}\pi + 2k\pi$ then $y = -\frac{1}{2}\sqrt{3} - \sqrt{3} = -\frac{3}{2}\sqrt{3}$, if $x = -\frac{2}{3}\pi + 2k\pi$ then $y = \frac{1}{2}\sqrt{3} + \sqrt{3} = \frac{3}{2}\sqrt{3}$ and if $x = 2k\pi$ then $y = 0$. Therefore, the tangent line to the curve is horizontal at the points $\left(\pm\frac{2}{3}\pi + 2k\pi, \mp\sqrt{3}\right)$ and $(2k\pi, 0)$, where k is an integer (and at no other points).

b. If $y = x^{x^2}$, then logarithmic differentiation gives

$$\frac{dy}{dx} = x^{x^2} \frac{d}{dx}\left(x^2 \log x\right) = x^{x^2}(2x\log x + x) = x^{x^2+1}(2\log x + 1),$$

which is equal to zero if, and only if, $\log x = -\frac{1}{2}$, or $x = e^{-1/2}$, where $y = (e^{-1/2})^{e^{-1}} = e^{-1/(2e)}$. Thus, the curve has one horizontal tangent line, at the point $\left(e^{-1/2}, e^{-1/(2e)}\right)$.

Solution to question 8. — The parabola is defined by $x^2 = 4fy$ and the circle is defined by $x^2 + y^2 = f^2$. A line is tangent to the parabola at (x, y) and to the circle at (p, q) if, and only if,

$$x^2 = 4fy, \qquad p^2 + q^2 = f^2, \qquad \frac{x}{2f} = \frac{y - q}{x - p} \qquad \text{and} \qquad \frac{x}{2f} = -\frac{p}{q}.$$

The third and first equations give $2x^2 - 2px = 4fy - 4fq = x^2 - 4fq$, and the second equation gives

$$(x - p)^2 = p^2 - 4fq = f^2 - 4fq - q^2, \qquad \text{or} \qquad (x - p)^2 + (2f + q)^2 = 5f^2.$$

The fourth equation implies that

$$(x - p)^2 = \frac{p^2}{q^2}(2f + q)^2, \qquad \text{and hence} \qquad (2f + q)^2 = 5q^2.$$

Therefore, $2f + q = -q\sqrt{5}$, or

$$q = \frac{-2f}{1 + \sqrt{5}} = -\frac{f}{\tau}, \qquad \text{and} \qquad p^2 = f^2 - q^2 = \frac{f^2}{\tau}, \qquad \text{so} \qquad p = \pm\frac{f}{\sqrt{\tau}},$$

where τ is the positive solution of $x^2 = x + 1$ (the so-called golden section). Thus, the tangent lines have slope $\pm\sqrt{\tau}$, are defined by $y + f\tau = \pm x\sqrt{\tau}$, and make contact with the parabola at the points $(\pm 2f\sqrt{\tau}, f\tau)$.

Solution to question 9. — First observe that if $y = e^{-3x}g(x)$, then

$$\frac{dy}{dx} = e^{-3x}(-3)g(x) + e^{-3x}g'(x) = e^{-3x}\big(-3g(x) + g'(x)\big).$$

Using this to compute and then carefully simplify the first four derivatives of $f(x) = e^{-3x}x^2$, gives

$$\begin{aligned}
f'(x) &= e^{-3x}(-3x^2 + 2x),\\
f''(x) &= e^{-3x}\big(-3(-3x^2 + 2x) - 6x\big) = e^{-3x}(9x^2 - 6 \cdot 2x + 2),\\
f^{(3)}(x) &= e^{-3x}\big(-3(9x^2 - 6 \cdot 2x + 2) + 18x - 6 \cdot 2\big) = e^{-3x}(-3)(9x^2 - 6 \cdot 3x + 3 \cdot 2) \qquad \text{and}\\
f^{(4)}(x) &= e^{-3x}(-3)\big(-3(9x^2 - 6 \cdot 3x + 3 \cdot 2) + 18x - 6 \cdot 3\big) = e^{-3x}(-3)^2(9x^2 - 6 \cdot 4x + 4 \cdot 3).
\end{aligned}$$

This suggests the pattern

$$f^{(n)}(x) = e^{-3x}(-3)^{n-2}\big(9x^2 - 6nx + n(n-1)\big),$$

which is readily verified in the four cases displayed. Since

$$\begin{aligned}
\frac{d}{dx}\Big\{e^{-3x}(-3)^{n-2}\big(9x^2 - 6nx + n(n-1)\big)\Big\} &= e^{-3x}(-3)^{n-2}\big(-3(9x^2 - 6nx + n(n-1)) + 2 \cdot 9x - 6n\big)\\
&= e^{-3x}(-3)^{n-1}\big(9x^2 - 6(n+1)x + (n+1)n\big),
\end{aligned}$$

it follows that the pattern will continue for all higher derivatives.

Solution to question 10. — a. Where u and v are differentiable functions of x, so is $y = uv$, and

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Proof. — First observe that

$$\frac{dy}{dx} = \lim_{x'\to x} \frac{y'-y}{x'-x} = \lim_{x'\to x} \frac{u'v' - uv' + uv' - uv}{x'-x} = \lim_{x'\to x}\left\{\frac{u'-u}{x'-x}\cdot v' + u\cdot\frac{v'-v}{x'-x}\right\}.$$

Now by the definition of the derivative,

$$\lim_{x'\to x}\frac{u'-u}{x'-x} = \frac{du}{dx} \qquad \text{and} \qquad \lim_{x'\to x}\frac{v'-v}{x'-x} = \frac{dv}{dx}.$$

Also, the continuity of differentiable functions implies that $\lim_{x'\to x} v' = v$. Therefore,

$$\frac{dy}{dx} = \lim_{x'\to x}\left\{\frac{u'-u}{x'-x}\cdot v' + u\cdot\frac{v'-v}{x'-x}\right\} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

□

b. If p and q are positive integers with no common factor, and $y = x^{-\frac{p}{q}}$, then $\frac{dy}{dx} = -\frac{p}{q}x^{-\frac{p+q}{q}}$.

Proof. — If $y = x^{-\frac{p}{q}}$ and $y' = x'^{-\frac{p}{q}}$, then

$$\frac{dy}{dx} = \lim_{x'\to x}\frac{y'-y}{x'-x} = \lim_{x'\to x}\frac{x'^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{x'-x}.$$

Let $z = x^{\frac{1}{q}}$ and $z' = x'^{\frac{1}{q}}$; then $z' \to z$ as $x' \to x$ and

$$\begin{aligned}\frac{x'^{-\frac{p}{q}} - x^{-\frac{p}{q}}}{x'-x} &= \frac{z'^{-p} - z^{-p}}{z'^q - z^q} = \frac{-1}{z'^p z^p}\cdot\frac{z'^p - z^p}{z'^q - z^q}\\ &= \frac{-1}{z'^p z^p}\cdot\frac{(z'-z)(z'^{p-1} + z'^{p-2}z + \cdots + z'z^{p-2} + z^{p-1})}{(z'-z)(z'^{q-1} + z'^{q-2}z + \cdots + z'z^{q-2} + z^{q-1})}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{z'\to z}\frac{-1}{z'^p z^p}\cdot\frac{z'^{p-1} + z'^{p-2}z + \cdots + z'z^{p-2} + z^{p-1}}{z'^{q-1} + z'^{q-2}z + \cdots + z'z^{q-2} + z^{q-1}}\\ &= \frac{-1}{z^{2p}}\cdot\frac{pz^{p-1}}{qz^{q-1}} = -\frac{p}{q}z^{-(p+q)}\\ &= -\frac{p}{q}x^{-\frac{p+q}{q}}.\end{aligned}$$

□

Solutions to 'Another sample test'

Solution to question 1. — a. If $f(x) = x/(2x+1)$, then

$$f(x') - f(x) = \frac{x'}{2x'+1} - \frac{x}{2x+1} = \frac{x'(2x+1) - (2x'+1)x}{(2x'+1)(2x+1)} = \frac{x'-x}{(2x'+1)(2x+1)},$$

and so

$$f'(x) = \lim_{x'\to x}\frac{f(x')-f(x)}{x'-x} = \lim_{x'\to x}\frac{1}{(2x'+1)(2x+1)} = \frac{1}{(2x+1)^2}.$$

b. The tangent line to the graph of f has y intercept 2 where

$$\frac{\frac{x}{2x+1} - 2}{x - 0} = \frac{1}{(2x+1)^2}, \qquad \text{or} \qquad x = (2x+1)(x - 2(2x+1)) = -(2x+1)(3x+2).$$

Equivalently, $0 = 6x^2 + 8x + 2 = 2(x+1)(3x+1)$. Since $f(-1) = 1$ and $f(-\frac{1}{3}) = -1$, it follows that the tangent to the graph of f at the points $(-1, 1)$ and $(-\frac{1}{3}, -1)$ have y intercept 2 (and there are no other such points on the graph of f).

Solution to question 2. — a. If $0 < \alpha < \beta$ then $\log\beta - \log\alpha$ is the area of the region defined by $\alpha \leqslant x \leqslant \beta$ and $0 \leqslant y \leqslant 1/x$. The region contains a rectangle of base $\beta - \alpha$ and height $1/\beta$, and is contained in a rectangle of base $\beta - \alpha$ and height $1/\alpha$, so

$$\frac{\beta-\alpha}{\beta} < \log\beta - \log\alpha < \frac{\beta-\alpha}{\alpha}, \qquad \text{or} \qquad \frac{1}{\beta} < \frac{\log\beta - \log\alpha}{\beta-\alpha} < \frac{1}{\alpha}.$$

Therefore,

$$\lim_{\alpha\to\beta^-}\frac{\log\beta - \log\alpha}{\beta-\alpha} = \frac{1}{\beta} \qquad \text{and} \qquad \lim_{\beta\to\alpha^+}\frac{\log\beta - \log\alpha}{\beta-\alpha} = \frac{1}{\alpha},$$

so by the definition of the derivative,

$$\frac{d}{dx}\left\{\log(x)\right\} = \frac{1}{x}, \qquad \text{provided} \quad x > 0.$$

b. Since the logarithm is continuous with a continuous inverse (by Part a), and

$$\lim_{t\to 0}\frac{\log(1+t)}{t} = \frac{d}{dx}\left\{\log(x)\right\}\Big|_{x=1} = 1, \qquad \text{it follows that} \qquad \lim_{t\to 0}(1+t)^{1/t} = e,$$

where e is the unique solution of the equation $\log(x) = 1$. Therefore,

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \left(\lim_{t\to 0}(1+t)^{1/t}\right)^x = e^x, \qquad \text{where} \qquad x = nt.$$

Solution to question 3. — If $f(x) < g(x)$ for $x \neq a$ and $f(a) = g(a)$, then $f(x) - f(a) < g(x) - g(a)$ for $x \neq a$. Therefore,

$$\frac{f(x)-f(a)}{x-a} > \frac{g(x)-g(a)}{x-a} \quad \text{if} \quad x < a \qquad \text{and} \qquad \frac{f(x)-g(a)}{x-a} < \frac{g(x)-g(a)}{x-a} \quad \text{if} \quad x > a.$$

Since f and g are differentiable at a it follows that

$$f'(a) = \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a} \geqslant \lim_{x\to a^-}\frac{g(x)-g(a)}{x-a} = g'(a),$$

and

$$f'(a) = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a} \leqslant \lim_{x\to a^+}\frac{g(x)-g(a)}{x-a} = g'(a),$$

which implies that $f'(a) = g'(a)$.

Solution to question 4. — a. If

$$u = \left(\sqrt[3]{r^5} - \frac{4}{\sqrt[5]{r^3}} + e^\pi\right)\sin(\pi^e) = \left(r^{5/3} - 4r^{-3/5} + e^\pi\right)\sin(\pi^e),$$

then

$$\frac{du}{dr} = \left(\tfrac{5}{3}r^{2/3} + \tfrac{12}{5}r^{-8/5}\right)\sin(\pi^e).$$

b. If $p = \tan\left(q^3\sin(q) + \sqrt{\cos(\ln q)}\right)$, then

$$\frac{dp}{dq} = \sec^2\left(q^3\sin(q) + \sqrt{\cos(\ln q)}\right)\left(3q^2\sin(q) + q^3\cos(q) - \frac{\sin(\ln q)}{2q\sqrt{\cos(\ln q)}}\right).$$

c. If $f(x) = \dfrac{\sec x + \log_2(\cos x)}{\sqrt{x^3 - x^2 + 7}}$ then, since $\log_2 = (\ln 2)^{-1}\log$,

$$f'(x) = \frac{\sec(x)\tan(x) - (\ln 2)^{-1}\tan(x)}{\sqrt{x^3 - x^2 + 7}} - \frac{\left(\sec(x) + \log_2(\cos x)\right)(3x^2 - 2x)}{2(x^3 - x^2 + 7)^{3/2}}.$$

d. If

$$y\sin(x^2y) = 1-\cos(2x+3y), \qquad \textit{i.e.} \qquad y\sin(x^2y)+\cos(2x+3y)=1,$$

then implicit differentiation gives

$$\frac{dx}{dy} = -\frac{\sin(x^2y)+x^2y\cos(x^2y)-3\sin(2x+3y)}{2xy^2\cos(x^2y)-2\sin(2x+3y)}.$$

e. If $y=\tan\big((\sin x)^{\cot(x)}\big)$, then the chain rule and logarithmic differentiation give

$$\begin{aligned}\frac{dy}{dx} &= \sec^2\big((\sin x)^{\cot(x)}\big)(\sin x)^{\cot(x)}\,\frac{d}{dx}\big\{\cot(x)\ln(\sin x)\big\}\\ &= \sec^2\big((\sin x)^{\cot(x)}\big)(\sin x)^{\cot(x)}\big(-\csc^2(x)\ln(\sin x)+\cot^2(x)\big).\end{aligned}$$

Solution to question 5. — Notice that the given equation can be written as $r^4-12r^2-36x^2=0$, where $r^2=x^2+y^2$ is the square of the distance between (x,y) and the origin, and that this implies that $12\leqslant r^2\leqslant 48$ for any point on $\mathscr{H}$ besides the origin (so the origin is an isolated point).

a. Differentiating implicitly with respect to x gives

$$\frac{dy}{dx} = -\frac{4xr^2-24x-72x}{4yr^2-24y} = \frac{x(24-r^2)}{y(r^2-6)}.$$

b. A point on $\mathscr{H}$ where $x=y$ satisfies $0=4x^4-60x^2=4x^2(x^2-15)$. Since the origin is an isolated point, there is no tangent to the curve at the origin. At the points $(\pm\sqrt{15},\pm\sqrt{15})$, $r^2=30$ and

$$\frac{dy}{dx} = \frac{24-30}{30-6} = -\frac{1}{4},$$

so the tangent lines in are defined, respectively, by $y\mp\sqrt{15} = -\frac{1}{4}(x\mp\sqrt{15})$, or $x+4y=\pm5\sqrt{15}$.

c. The tangent line to $\mathscr{H}$ is horizontal where $\frac{dy}{dx}=0$, which is where

$$x(24-r^2)=0, \qquad \text{and} \qquad y(r^2-6)\neq 0.$$

If $x=0$ on $\mathscr{H}$ then $y^4-12y^2=0$, or $y^2(y^2-12)=0$, and since the origin is isolated the only possibility is where $y^2=12$. This gives two points, $(0,\pm2\sqrt{3})$, where the tangent line is horizontal. If $r^2=24$ on $\mathscr{H}$ then $24\cdot 12=36x^2$, or $x^2=8$, and so $y^2=16$. This gives four more points, $(2\sqrt{2},4)$, $(-2\sqrt{2},4)$, $(-2\sqrt{2},-4)$ and $(2\sqrt{2},-4)$, where the tangent line is horizontal. There are no other points on $\mathscr{H}$ at which the tangent line is horizontal.

d. The tangent line to $\mathscr{H}$ is vertical where $\frac{dx}{dy}=0$, which is where

$$y(r^2-6)=0, \qquad \text{and} \qquad x(24-r^2)\neq 0.$$

There is no point on $\mathscr{H}$ where $r^2=6$ since every point on the curve besides the origin satisfies $12\leqslant r^2\leqslant 48$. If $y=0$ on $\mathscr{H}$, then $x^4-12x^2=36x^2$, or $x^2(x^2-48)=0$, and since the origin is an isolated point on the curve, the tangent to the curve is vertical where $x^2=48$. Hence there are two points, $(\pm4\sqrt{3},0)$, at which the tangent line to $\mathscr{H}$ is vertical.

Solution to question 6. — If $y=\tan^2(\gamma x)$ then $1+y=\sec^2(\gamma x)$. Differentiating twice with respect to x gives

$$\frac{dy}{dx} = 2\gamma\tan(\gamma x)\sec^2(\gamma x),$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= 2\gamma^2\big\{\sec^4(\gamma x)+2\tan^2(\gamma x)\sec^2(\gamma x)\big\} = 2\gamma^2\big\{(1+y)^2+2y(1+y)\big\}\\ &= 2\gamma^2(1+y)(1+3y).\end{aligned}$$

Therefore,

$$\frac{d^2y}{dx^2} = 8(1+\alpha y)(1+\beta y)$$

if, and only if, $\alpha=1$ and $\beta=3$, or vice versa, and $\gamma=\pm2$.

Solution to question 7. — a. If $f(x)=x(\log x)^2$ then

$$f'(x) = (\log x)^2+2\log x = (\log x)(\log x+2),$$

which is zero if, and only if, either $\log x=0$, *i.e.*, $x=1$, or else $\log x=-2$, *i.e.*, $x=e^{-2}$. Therefore, the tangent line to the graph of f is horizontal if x is 1 or e^{-2}.

b. If $f(x)=(x-2)^2(x^2+2x-12)^3$ then

$$\begin{aligned}f'(x) &= 2(x-2)(x^2+2x-12)^3+(x-2)^2 3(x^2+2x-12)^2(2x+2)\\ &= 2(x-2)(x^2+2x-12)\big(x^2+2x-12+3(x-2)(x+1)\big)\\ &= 2(x-2)(x^2+2x-12)(4x^2-x-18)\\ &= 2(x-2)(x^2+2x-12)(4x-9)(x+2).\end{aligned}$$

The tangent line to the graph of f is horizontal where $f'(x)=0$. Since $x^2+2x-12=(x+1)^2-13$, it follows that the graph of f has horizontal tangent lines where $x=\pm2$, $\frac{9}{4}$ or $-1\pm\sqrt{13}$.

Solution to question 8. — a. If the position of the particle is given by $x=(2t-1)^3(t-2)^6$, for $t\geqslant 0$, then its velocity is

$$v=\frac{dx}{dt} = 6(2t-1)^2(t-2)^6+6(2t-1)^3(t-2)^5 = 18(2t-1)^2(t-2)^5(t-1).$$

The particle is at rest when its velocity is zero; *i.e.*, when t is $\frac{1}{2}$, 1, 2.

b. The particle is moving to the left when its velocity is negative and perhaps zero at isolated instants. Since $(2t-1)^2$ is never negative, and $(t-2)^5$ and $t-1$ have opposite signs only if $1<t<2$, it follows that the particle is moving to the left on the time interval $(1,2)$.

c. The particle moves to the right if $0<t<1$, to the left if $1<t<2$ and again to the right if $2<t<3$, so the distance travelled by the particle during the first three seconds is equal to

$$\big\{s(1)-s(0)\big\}+\big\{s(1)-s(2)\big\}+\big\{s(3)-s(2)\big\} = \big\{1-(-64)\big\}+\big\{1-0\big\}+\big\{125-0\big\} = 191 \text{ metres}.$$

Solution to question 9. — If

$$y=a+bx+cx^2+dx^3+x^4, \qquad \text{then} \qquad \frac{dy}{dx} = b+2cx+3dx^2+4x^3.$$

The curve is tangent to $y=-3x-2$ where $x=-1$ if, and only if, $y=1$ and $\frac{dy}{dx}=-3$ where $x=-1$, so

$$a-b+c-d+1=1 \qquad \text{and} \qquad b-2c+3d-4=-3.$$

The curve is tangent to $y=6-7x$ where $x=1$ if, and only if, $y=-1$ and $\frac{dy}{dx}=-7$ where $x=1$, so

$$a+b+c+d+1=-1 \qquad \text{and} \qquad b+2c+3d+4=-7.$$

Adding the first and third equations gives $2a+2c+2=0$, or $a+c=-1$. Subtracting the fourth second equation from the fourth equation gives $4c+8=-4$, or $c=-3$, and therefore $a=2$. Replacing a and c by their values in the first and second equations gives $-b-d=1$ and $b+3d=-5$. Adding these last two equations gives $2d=-4$, so $d=-2$ and $b=1$. Therefore, the curve in question is defined by

$$y=2+x-3x^2-2x^3+x^4.$$

Solution to question 10. — a. Where y is a differentiable function of x and z is a differentiable function of y, z is a differentiable function of x, and

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

Proof. — Where y is a differentiable function of x,

$$\frac{dy}{dx} = \lim_{x' \to x} \frac{y' - y}{x' - x}, \quad \text{or} \quad y' - y = \left(\frac{dy}{dx} + \varepsilon\right)(x' - x), \quad \text{in which } \varepsilon \to 0 \text{ as } x' \to x.$$

Likewise, where z is a differentiable function of y,

$$\frac{dz}{dy} = \lim_{y' \to y} \frac{z' - z}{y' - y}, \quad \text{or} \quad z' - z = \left(\frac{dz}{dy} + \eta\right)(y' - y), \quad \text{in which } \eta \to 0 \text{ as } y' \to y.$$

Combining these results gives (for $x' \neq x$),

$$z' - z = \left(\frac{dz}{dy} + \eta\right)\left(\frac{dy}{dx} + \varepsilon\right)(x' - x), \qquad \text{or} \qquad \frac{z' - z}{x' - x} = \left(\frac{dz}{dy} + \eta\right)\left(\frac{dy}{dx} + \varepsilon\right).$$

As $x' \to x$, $\varepsilon \to 0$ and $\eta \to 0$ (for $y' \to y$ by the continuity of differentiable functions). Therefore,

$$\frac{dz}{dx} = \lim_{x' \to x} \frac{z' - z}{x' - x} = \lim_{\eta \to 0}\left(\frac{dz}{dy} + \eta\right) \cdot \lim_{\varepsilon \to 0}\left(\frac{dy}{dx} + \varepsilon\right) = \frac{dz}{dy}\frac{dy}{dx}.$$ □

b. Subtracting the sum and difference identities of the sine function gives

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta.$$

If $x' = \alpha + \beta$ and $x = \alpha - \beta$, then $\alpha = \frac{1}{2}(x' + x)$ and $\beta = \frac{1}{2}(x' - x)$. Therefore,

$$\begin{aligned}\frac{d}{dx}\left\{\sin(x)\right\} = \lim_{x' \to x} \frac{\sin(x') - \sin(x)}{x' - x} &= \lim_{x' \to x}\left\{\cos\left(\tfrac{1}{2}(x' + x)\right) \cdot \frac{\sin\left(\frac{1}{2}(x' - x)\right)}{\frac{1}{2}(x' - x)}\right\} \\ &= \cos\left(\tfrac{1}{2}(x + x)\right) \cdot 1 \\ &= \cos(x).\end{aligned}$$ □