An old Test 3

Question 1. — A cone of height 7 cm and radius 2 cm is dropped point first into a tall cylinder of radius 4 cm. At the instant the cone is completely submerged, it is falling at a rate of $\frac{4}{3}$ cm/s. At what rate is the water level rising at this time?

Question 2. — Find the global extrema of $f(x)=(x+3)^{1/3}(x-1)^{2/3}$ on the closed interval $\left[-3, \frac{5}{3}\right]$.

Question 3. — a. Find $g(t)$ if $g''(t)=3e^{-t}-12\cos(2t)$, $g'(0)=-1$ and $g(0)=2$.

b. Prove that if f is differentiable on $\mathbb{R}$ then between any two zeros of f there is a zero of $f'+f$.

c. Prove that if $\dfrac{d^2y}{dx^2}+4y=4\dfrac{dy}{dx}$ for all $x\in\mathbb{R}$, then there are $\alpha,\beta\in\mathbb{R}$ such that $y=e^{2x}(\alpha x+\beta)$.

Question 4. — Find the intervals of monotonicity and all extreme values of $y=\sin(2x)-2\sin(x)$.

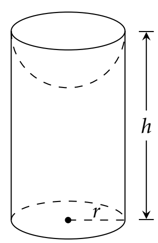
Question 5. — Sketch the graph of $y=(x+2)e^{1/x}$. Indicate clearly the domain, and all intercepts, asymptotes, intervals of monotonicity and concavity, extreme values and points of inflection.

Question 6. — Sketch the graph of $f(x)=(x^2-1)^{2/3}$, given

$$f'(x)=\frac{4x}{3(x^2-1)^{1/3}} \qquad \text{and} \qquad f''(x)=\frac{4(x^2-3)}{9(x^2-1)^{4/3}}.$$

Indicate clearly the domain, and all intercepts, asymptotes, intervals of monotonicity and concavity, extreme values and points of inflection.

Question 7. — A closed cylindrical tank with a flat bottom and an inverted hemispherical top is to have a volume of 13π. Find the dimensions that will minimize the cost of the metal to make the tank, and state the resulting ratio h/r of height to radius.



Question 8. — Evaluate each of the following integrals.

a. $\displaystyle\int_0^3 |2x-x^2|\,dx$ b. $\displaystyle\int \frac{(x+2)^2}{\sqrt{x}}\,dx$ c. $\displaystyle\int_{\frac{2}{3}\pi}^{\frac{7}{6}\pi} \frac{\sin^2(\vartheta)}{1+\cos(2\vartheta)}\,d\vartheta$

d. $\displaystyle\int_0^3 \left(|x-1|+\sqrt{4x-x^2}\right)dx$

Question 9. — a. Evaluate each definite integral as a limit of Riemann sums:

$$\int_1^4 (5x^2-4x+2)\,dx; \qquad \int_{-1}^1 5^{2x-1}\,dx.$$

b. Evaluate the limit $\displaystyle\lim_{k\to\infty}\left\{\frac{1}{4k+3}+\frac{1}{4k+6}+\frac{1}{4k+9}+\cdots+\frac{1}{13k}\right\}$.

Question 10. — Find the intervals of concavity, and the x coordinates of all points of inflection, of the function defined by

$$y=\int_1^{\sqrt{x}} \frac{2z^3}{\sqrt{z^6+4}}\,dz, \qquad \text{for} \quad x>0.$$

Another old Test 3

Question 1. — A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of $\frac{1}{5}\,\mathrm{m}^3$/min, how fast is the water level rising when the water is 30 cm deep?

Question 2. — Find the largest and smallest values of $f(x)=x^2e^{-x}$ on the closed interval $[1,4,]$.

Question 3. — a. Prove that $f(x)=3x^5+5x^3-30x+1$ has exactly three real roots.

b. A number p is a **fixed point** of a function f if $f(p)=p$. Prove that if f is everywhere differentiable, $f'(x)\neq 1$ for all real numbers x, and there are real numbers $a<b$ such that $a\leqslant f(x)\leqslant b$ whenever $a\leqslant x\leqslant b$, then f has exactly one fixed point.

c. Suppose that $|f''|\leqslant 1$ on $[a,b]$, where $a<b$, $f(a)=0$, $f(b)\neq 0$ and $|f'(b)|\geqslant 1$. Let c be the x intercept of the tangent line to the graph of f at the point $(b,f(b))$. Prove that $|a-c|<(a-b)^2$.

Question 4. — Find the intervals of monotonicity and all extreme values of $y=(\log x)^2/x$.

Question 5. — Sketch the graph of $f(x)=(2x+3)(x-3)^2/x^3=2-9/x+27/x^3$. Indicate clearly the domain, and all intercepts, asymptotes, intervals of monotonicity and concavity, extreme values and points of inflection.

Question 6. — Sketch the graph of $y=x\sqrt{4-x^2}$, given $\dfrac{dy}{dx}=\dfrac{2(2-x^2)}{\sqrt{4-x^2}}$ and $\dfrac{d^2y}{dx^2}=\dfrac{2x(x^2-6)}{(4-x^2)^{3/2}}$. Indicate clearly the domain, and all intercepts, asymptotes, intervals of monotonicity and concavity, extreme values and points of inflection.

Question 7. — A right circular cylinder is inscribed in a sphere of radius r. Solve **one** (your choice) of the following problems.

- Find the largest possible volume of the cylinder.
- Find the largest possible surface area (top and bottom included) of the cylinder.

Question 8. — Evaluate each of the following integrals.

a. $\displaystyle\int_{-1}^1 \frac{(2^x-3^x)^2}{6^x}\,dx$ b. $\displaystyle\int \frac{(y-1)^3}{y^3-y^2}\,dy$ c. $\displaystyle\int \frac{\sin(\vartheta)}{1+\sin(\vartheta)}\,d\vartheta$ d. $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \left\{\sqrt{2}+\sqrt{2-x^2}-\sqrt{4-x^2}\right\}dx$

Question 9. — a. Evaluate each definite integral as a limit of Riemann sums:

$$\int_{-1}^3 (x^3-4x+2)\,dx; \qquad \int_{\frac{1}{3}\pi}^{\frac{7}{6}\pi} \cos(3x)\,dx.$$

b. Express the limit $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \ln(3-k/n)$ as a definite integral with:

i. lower limit 0; ii. upper limit -4; iii. integrand $\log(4-x)$.

Question 10. — a. Verify the integral formula

$$\int \frac{dx}{x^4\sqrt{2x^2+1}}=\frac{(4x^2-1)\sqrt{2x^2+1}}{3x^3}, \qquad \text{and use it to evaluate} \qquad \int_1^2 \frac{dx}{x^4\sqrt{2x^2+1}}.$$

b. Give a formula for a function f which satisfies $\displaystyle\int_1^x f(t)\,dt=\frac{\log(x)}{x}-\int_1^{x^2}\frac{f(\sqrt{t})}{t}\,dt$ for $x>0$.

Solutions to an old Test 3

Solution to Question 1. — Let z be the distance the tip of the cone has moved downward (after it goes into the water), let y be the distance the water level in the cylinder has risen, and let h be the height of the submerged portion of the cone (all measured in centimetres). Then $h = y + z$, and the radius of the submerged portion of the cone is $r = \frac{2}{7}h$ by similarity. The volume of the submerged portion of the cone is given by

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{4}{147}\pi h^3, \qquad \text{and so} \qquad \frac{dV}{dt} = \tfrac{4}{49}\pi h^2 \frac{dh}{dt}.$$

Since the volume of the water in the cylinder does not change, the volume of the submerged portion of the cone is also equal to

$$V = \pi 4^2 y = 16\pi y, \qquad \text{and so} \qquad \frac{dV}{dt} = 16\pi \frac{dy}{dt}.$$

From $h = y + z$, it follows that

$$16\pi \frac{dy}{dt} = \tfrac{4}{49}\pi h^2\left(\frac{dy}{dt} + \frac{dz}{dt}\right), \qquad \text{or} \qquad \frac{dy}{dt} = \frac{h^2}{196 - h^2}\frac{dz}{dt}.$$

At the instant when the cone is completely submerges, $h = 7$ and $\frac{dz}{dt} = \frac{4}{3}$, so

$$\frac{dy}{dt} = \frac{49}{196 - 49} \cdot \frac{4}{3} = \frac{4}{9}.$$

Therefore, the water level in the cylinder is rising at a rate of $\frac{4}{9}$ cm/s at the instant when the cone is completely submerged.

Solution to Question 2. — The function defined by $f(x) = (x+3)^{1/3}(x-1)^{2/3}$, is continuous on $\mathbb{R}$, and

$$f'(x) = \tfrac{1}{3}(x+3)^{-2/3}(x-1)^{2/3} + \tfrac{2}{3}(x+3)^{1/3}(x-1)^{-1/3} = \tfrac{1}{3}(x+3)^{-2/3}(x-1)^{-1/3}(3x+5),$$

which is zero if $x = -\frac{5}{3}$ and is undefined if $x = -3$ or $x = 1$, so the critical numbers of f in the open interval $(-3, \frac{5}{3})$ are $-\frac{5}{3}$ and 1. Comparing $f(-3) = 0$, $f\left(-\frac{5}{3}\right) = \frac{4}{3}\sqrt[3]{4}$, $f(1) = 0$ and $f\left(\frac{5}{3}\right) = \frac{2}{3}\sqrt[3]{7}$, reveals that the extreme values of f on the interval $[-3, \frac{5}{3}]$ are 0 and $\frac{4}{3}\sqrt[3]{4}$.

Solution to Question 3. — a. If $g''(t) = 3e^{-t} - 12\cos(2t)$ and $g'(0) = -1$ then the mean value theorem implies that $g'(t) = -3e^{-t} - 6\sin(2t) + 2$. Further, if $g(0) = 2$, the mean value theorem implies that $g(t) = 3e^{-t} + 3\cos(2t) + 2t - 4$.

b. Suppose that $a < b$ and that $f(a) = f(b) = 0$. If $g(x) = e^x f(x)$, then g is differentiable on $\mathbb{R}$ and $g(a) = g(b) = 0$, so Rolle's Theorem implies that there is a real number ξ such that $a < \xi < b$ and $g'(\xi) = 0$, or equivalently $e^{\xi}\,(f(\xi) + f'(\xi)) = 0$. Since $e^{\xi} \neq 0$, it follows that $f(\xi) + f'(\xi) = 0$, as required.

c. If y is a solution of the given equation and $z = e^{-2x}y$ then

$$\frac{dz}{dx} = e^{-2x}\left(\frac{dy}{dx} - 2y\right) \qquad \text{and} \qquad \frac{d^2z}{dx^2} = e^{-2x}\left(\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y\right) = 0$$

for all real values of x. By the mean value theorem, there is a real number α such that $\frac{dz}{dx} = \alpha$. By another application of the mean value theorem, there is a real number β such that $z = \alpha x + \beta$, or equivalently $e^{-2x}y = \alpha x + \beta$. Therefore, $y = e^{2x}(\alpha x + \beta)$, as required.

Solution to Question 4. — If $y = \sin(2x) - 2\sin(x)$ then

$$\frac{dy}{dx} = 2\cos(2x) - 2\cos(x) = 2(2\cos^2(x) - \cos(x) - 1) = 2(2\cos(x) + 1)(\cos(x) - 1),$$

which is everywhere defined and zero if, and only if, either $\cos(x) = -\frac{1}{2}$, or else $\cos(x) = 1$. The solutions of $\cos(x) = -\frac{1}{2}$ are $\pm\frac{2}{3}\pi + 2\pi k$, where k is an integer, and the solutions of $\cos(x) = 1$ are $2\pi k$, where k is an integer. Notice that $\cos(x) - 1 \leqslant 0$ for all real values of x, so if $-\frac{2}{3}\pi + 2\pi k < x < \frac{2}{3}\pi + 2\pi k$ then $\cos(x) > -\frac{1}{2}$ so $\frac{dy}{dx} \leqslant 0$, and if $\frac{2}{3}\pi + 2\pi k < x < \frac{5}{3}\pi$ then $\cos(x) < -\frac{1}{2}$, so $\frac{dy}{dx} > 0$. Therefore, y is increasing on the intervals $\left[-\frac{2}{3}\pi + 2\pi k, \frac{2}{3}\pi + 2\pi k\right]$, where k is an integer, and y is decreasing on the intervals $\left[\frac{2}{3}\pi + 2\pi k, \frac{5}{3}\pi + 2\pi k\right]$, where k is an integer. The local and global minimum value of y is $-\frac{3}{2}\sqrt{3}$, which occurs where $x = \frac{2}{3}\pi + 2\pi k$ and k is an integer. The local and global maximum value of y is $\frac{3}{2}\sqrt{3}$, which occurs where $x = -\frac{2}{3}\pi + 2\pi k$ and k is an integer.

Solution to Question 5. — The domain is $\mathbb{R} \setminus \{0\}$, $y \to 0$ as $x \to 0^-$ and $y \to \infty$ as $x \to 0^+$, so the line defined by $x = 0$ is a vertical asymptote of graph, but from the right only. From the left, the graph approaches, but does not include, the origin. The graph no y-intercept (as 0 is not in the domain), and the x-intercept of the graph is -2. Observe that

$$\lim_{x\to\pm\infty}\left\{y - x\right\} = \lim_{x\to\pm\infty}\left\{2e^{1/x} + x(e^{1/x} - 1)\right\} = 2 + \lim_{t\to 0^{\pm}} \frac{e^t - 1}{t} = 2 + e^0 = 3,$$

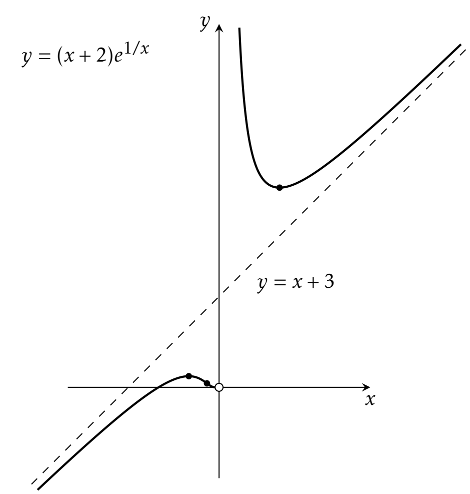
where $t = 1/x$. So the line defined by $y = x + 3$ is the oblique asymptote of the graph. Now

$$\frac{dy}{dx} = e^{1/x}\left(1 - x^{-1} - 2x^{-2}\right) = e^{1/x}x^{-2}(x^2 - x - 2) = e^{1/x}x^{-2}(x+1)(x-2),$$

which is zero if $x = -1$ or $x = 2$, positive if $x < -1$ or $x > 2$ and negative if $-1 < x < 0$ or $0 < x < 2$. Therefore, y is increasing on $(-\infty, -1]$ and on $[2, \infty)$, and decreasing on $[-1, 0)$ and on $(0, 2]$. So e^{-1} is a local maximum value of y and $4e^{1/2}$ is a local minimum value of y. Next,

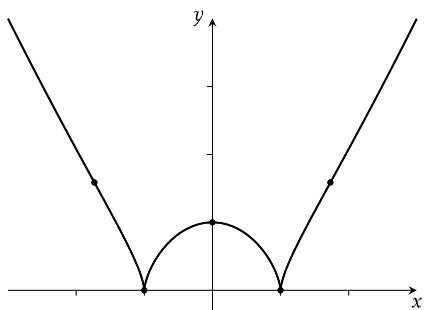
$$\frac{d^2y}{dx^2} = e^{1/x}\left(x^{-2} + 4x^{-3} - x^{-2} + x^{-3} + 2x^{-4}\right) = e^{1/x}x^{-4}(5x + 2),$$

which is zero if $x = -\frac{2}{5}$, negative if $x < -\frac{2}{5}$ and positive if $-\frac{2}{5} < x < 0$ or $x > 0$. Therefore, the graph is concave on the interval $\left(-\infty, -\frac{2}{5}\right]$, convex on the intervals $\left[-\frac{2}{5}, 0\right)$ and $(0, \infty)$, and has a point of inflection at $\left(-\frac{2}{5}, \frac{8}{5}e^{-5/2}\right)$. In the following sketch, the oblique asymptote is drawn as a dashed line and the points of interest are emphasized.



Solution to Question 6. — The domain of f is $\mathbb{R}$, on which f is continuous. The intercepts of the graph of f are $(0,1)$ and $(\pm1,0)$. Since $f(x) \geqslant 0$ for all real values of x, $f(\pm1) = 0$ is the global minimum value of f. Also, $f(x) = x^{4/3}(1-1/x^2)^{2/3}$, which implies that $f(x) \to \infty$ as $x \to \pm\infty$, and that the graph of f has no asymptotes. The derivative $f'(x)$ is equal to zero if $x = 0$, is undefined if $x = \pm1$, is negative if $x < -1$ or $0 < x < 1$ and is positive if $-1 < x < 0$ or $x > 1$. So f is increasing on $[-1,0]$ and on $[1,\infty]$, and is decreasing on $(-\infty,-1)$ and on $(0,1)$, with local (and global) minima (already noted) at $(\pm1,0)$ and a local maximum at $(0,1)$. Since $f'(x) \to -\infty$ as $x \to \pm1^-$ and since $f'(x) \to \infty$ as $x \to \pm1^+$, the graph of f has vertical cusps at $(\pm1,0)$. Next, $f''(x)$ is equal to zero if $x = \pm\sqrt{3}$, is undefined if $x = \pm1$, is negative if $|x| < \sqrt{3}$ and $x \neq \pm1$, and is positive if $|x| > \sqrt{3}$. So the graph of f is concave up on $(-\infty,-\sqrt{3})$ and on $(\sqrt{3},\infty)$, is concave down on $(-\sqrt{3},-1)$, on $(-1,1)$ and on $(1,\sqrt{3})$, and has points of inflection at $(\pm\sqrt{3},\sqrt[3]{4})$. Below is a sketch of the graph of f, with the points of interest emphasized.

$$y = (x^2-1)^{2/3}$$



Solution to Question 7. — The volume of the tank is given by

$$13\pi = \pi r^2 h - \tfrac{2}{3}\pi r^3, \qquad \text{and so} \qquad h = 13r^{-2} + \tfrac{2}{3}r.$$

The cost of the tank (assuming its parts are made of equally costly material) is proportional to its surface area, S, which is given by

$$S = 3\pi r^2 + 2\pi r h = 3\pi r^2 + 2\pi r(13r^{-2} + \tfrac{2}{3}r) = \tfrac{13}{3}\pi(r^2 + 6r^{-1})$$

for $r > 0$. Now

$$\frac{dS}{dr} = \tfrac{13}{3}\pi(2r - 6r^{-2}) = \tfrac{26}{3}\pi r^{-2}(r^3 - 3)$$

has the unique critical number $\sqrt[3]{3}$ on $(0,\infty)$, on which

$$\frac{d^2S}{dr^2} = \tfrac{26}{3}\pi(1 + 6r^{-3})$$

is positive. By the Second Derivative Test (for global extrema), the least expensive can has radius $\sqrt[3]{3}$ and height $\frac{13}{3}\sqrt[3]{3} + \frac{2}{3}\sqrt[3]{3} = 5\sqrt[3]{3}$, so the ratio of height to radius is 5.

Solution to Question 8. — a. As $2x - x^2 = x(2-x)$ is positive if $0 < x < 2$ negative if $x < 0$ or $x > 2$,

$$\int_0^3 |2x - x^2|\,dx = \int_0^2 (2x - x^2)\,dx - \int_2^3 (2x - x^2)\,dx = \left(x^2 - \tfrac{1}{3}x^3\right)\Big|_0^2 - \left(x^2 - \tfrac{1}{3}x^3\right)\Big|_2^3 = \tfrac{8}{3}.$$

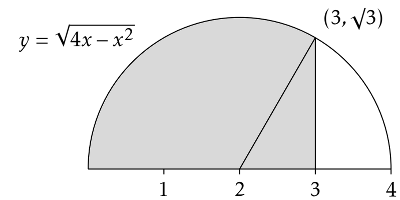
b. Expanding and dividing gives $(x+2)^2/\sqrt{x} = (x^2+4x+4)x^{-1/2} = x^{3/2} + 4x^{1/2} + 4x^{-1/2}$, and so

$$\int \frac{(x+2)^2}{\sqrt{x}}\,dx = \tfrac{2}{5}x^{5/2} + \tfrac{8}{3}x^{3/2} + 8x^{1/2} + C.$$

c. Since $1 + \cos(2\vartheta) = 2\cos^2(\vartheta)$, and $\tan^2(\vartheta) = \sec^2(\vartheta) - 1$, the integral is equal to

$$\tfrac{1}{2}\int_{\frac{2}{3}\pi}^{\frac{7}{6}\pi} (\sec^2(\vartheta) - 1)\,d\vartheta = \tfrac{1}{2}(\tan(\vartheta) - \vartheta)\Big|_{\frac{2}{3}\pi}^{\frac{7}{6}\pi} = \tfrac{2}{3}\sqrt{3} - \tfrac{1}{4}\pi.$$

d. The integral of $|x-1|$ on $[0,3]$ is the area of a triangle with base and height 1 plus the area of a triangle of base and height 2, which is equal to $\frac{1}{2} + 2 = \frac{5}{2}$. The graph of $y = \sqrt{4x - x^2} = \sqrt{4 - (x-2)^2}$ is an upper semicircle of radius 2 centred at $(2,0)$, and the integral of y on $[0,3]$ is the area of the shaded region in the figure, which consists of a circular sector of radius 2 and angle $\frac{2}{3}\pi$ and a triangle of base 1 and height $\sqrt{3}$, and is therefore equal to $\frac{4}{3}\pi + \frac{1}{2}\sqrt{3}$.



Therefore,

$$\int_0^3 \left(|x-1| + \sqrt{4x - x^2}\right)dx = \tfrac{5}{2} + \tfrac{1}{2}\sqrt{3} + \tfrac{4}{3}\pi.$$

Solution to Question 9. — a. If $[1,4]$ is divided into n subintervals of equal length then the width of each subinterval is $3/n$, the endpoints of the subintervals are $x_i = 1 + 3i/n$, and the values of $f(x) = 5x^2 - 4x + 2$ at the endpoints are $5\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2 = 3 + \frac{18i}{n} + \frac{45i^2}{n^2}$, for $i = 0,1,\ldots,n$. The Riemann sum obtained by evaluating f at the right endpoint of each subinterval is

$$\begin{aligned}\mathscr{R}_n &= \frac{3}{n}\sum_{i=1}^{n}\left\{3 + \frac{18i}{n} + \frac{45i^2}{n^2}\right\} = 9 + \frac{54}{n^2}\sum_{i=1}^{n} i + \frac{135}{n^3}\sum_{i=1}^{n} i^2 \\ &= 9 + \frac{54}{n^2}\cdot\frac{1}{2}n(n+1) + \frac{135}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1) = 9 + 27\left(1 + \frac{1}{n}\right) + 45\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right).\end{aligned}$$

Therefore,

$$\int_1^4 (5x^2 - 4x + 2)\,dx = \lim_{n\to\infty}\mathscr{R}_n = 9 + 27 + 45 = 81.$$

If the interval $[-1,1]$ is divided into k subintervals of equal length, then the length of each subinterval is $2/k$, the endpoints of the subintervals are $-1 + 2j/k$, and the values of $g(x) = 5^{2x-1}$ at these endpoints are $5^{2(-1+2j/k)-1} = 5^{-3}\cdot\left(5^{4/k}\right)^j$, for $j = 0,1,2,\ldots,k$. The Riemann sum obtained by evaluating g at the left endpoint of each subinterval is

$$\mathscr{L}_k = \frac{2}{k}\sum_{j=0}^{k-1} 5^{-3}\cdot\left(5^{4/k}\right)^j = \frac{2}{125k}\cdot\frac{5^4 - 1}{5^{4/k} - 1},$$

by the factorization of a difference of powers. If $x = 5^{4/k}$ then $4/k = \log(x)/\log(5)$ and $x \to 1$ as $k \to \infty$. Therefore,

$$\int_{-1}^{1} 5^{2x-1}\,dx = \lim_{k\to\infty} \mathscr{L}_k = \lim_{x\to 1}\left\{\frac{5^4-1}{250\log(5)}\cdot\frac{\log(x)}{x-1}\right\} = \frac{312}{125\log(5)}$$

b. Since $13k - 4k = 3\cdot 3k$, the expression in the limit is

$$\sum_{j=1}^{3k}\frac{1}{4k+3j} = \frac{1}{k}\sum_{j=1}^{3k}\frac{1}{4+\frac{3}{k}\cdot j} = \frac{1}{3}\cdot\frac{9}{n}\sum_{j=1}^{n}\frac{1}{4+\frac{9}{n}\cdot j},$$

where $n = 3k$. The expression on the right is a one-third of the right endpoint Riemann sum of $1/x$ on the interval $[\,4,13\,]$ for a partition of the interval into n subintervals of equal length. Therefore,

$$\lim_{k\to\infty}\left\{\frac{1}{4k+3}+\frac{1}{4k+6}+\frac{1}{4k+9}+\cdots+\frac{1}{13k}\right\} = \frac{1}{3}\int_4^{13}\frac{dx}{x} = \frac{1}{3}\log\frac{13}{4}.$$

Solution to Question 10. — By the first fundamental theorem of calculus and the chain rule, the first derivative of the given function is

$$\frac{dy}{dx} = \frac{2(\sqrt{x})^3}{\sqrt{(\sqrt{x})^6+4}}\cdot\frac{1}{2\sqrt{x}} = \frac{x}{\sqrt{x^3+4}},$$

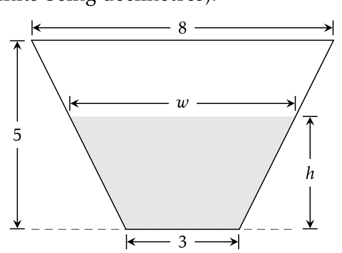
and the second derivative is

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x^3+4}} - \frac{3x^3}{2(x^3+4)^{3/2}} = \frac{8-x^3}{2(x^3+4)^{3/2}}.$$

The second derivative is positive if $0 < x < 2$ and negative if $x > 2$. So the function convex on $(0,2]$, concave on $[2,\infty)$, and has one point of inflection, where $x = 2$.

Solutions to another an old Test 3

Solution to Question 1. — A cross-section of the trough, with the shaded portion filled with water, is depicted below (the units being decimetres).



The area of the shaded cross section is $\frac{1}{2}h(w+h)$, and $w = 3+h$ by similarity, so the volume of the trough is

$$V = 100\cdot\tfrac{1}{2}h(h+6) = 50(h^2+6h), \qquad \text{so} \qquad \frac{dV}{dt} = 50(2h+6)\frac{dh}{dt} = 100(h+3)\frac{dh}{dt}.$$

When $h = 3$,

$$\frac{dV}{dt} = \frac{10^3}{5} = 200, \qquad \text{and so} \qquad 200 = 600\left.\frac{dh}{dt}\right|_{h=3}, \qquad \text{or} \qquad \left.\frac{dh}{dt}\right|_{h=3} = \frac{1}{3}.$$

Therefore, the water level is rising at the rate of $\frac{1}{3}$ decimetres (*i.e.*, $\frac{10}{3}$ centimetres, or $\frac{1}{30}$ metres) per minute when the water is 3 decimetres (*i.e.*, 30 centimetres) deep.

Solution to Question 2. — Since $f'(x) = 2xe^{-x} - x^2e^{-x} = e^{-x}x(2-x)$, for all real values of x, the only critical number of f in $(1,4)$ is 2. The pertinent values of f are

$$f(1) = \tfrac{1}{e}, \qquad f(2) = 4e^{-2} = \left(\tfrac{2}{e}\right)^2 \qquad \text{and} \qquad f(4) = 16e^{-4} = \left(\tfrac{2}{e}\right)^4.$$

Since $e < 4$, $\frac{1}{e} < \frac{1}{e}\cdot\frac{4}{e} = \left(\frac{2}{e}\right)^2$. Also $e > \frac{13}{5}$, and hence $e^3 > \left(\frac{13}{5}\right)^3 = \frac{169}{25}\cdot\frac{13}{5} > \frac{165}{26}\cdot\frac{13}{5} = \frac{33}{2} > 16$; therefore, $\left(\frac{2}{e}\right)^4 = \frac{16}{e^3}\cdot\frac{1}{e} < \frac{1}{e}$. Thus, the largest value of f on $[1,4]$ is $f(2) = 4e^{-2}$ and the smallest value of f on $[1,4]$ is $f(4) = 16e^{-4}$.

Solution to Question 3. — a. Since f is a polynomial function, it is differentiable (and therefore also continuous) on every interval of positive length. Since

$$f(-2) = -96-40+60+1 = -75 < 0, \qquad f(-1) = -3-5+30+1 = 23 > 0, \qquad f(0) = 1 > 0,$$
$$f(1) = 3+5-30+1 = -21 < 0 \qquad \text{and} \qquad f(2) = 96+40-60+1 = 77 > 0,$$

the intermediate value theorem implies that f has a root in each of the intervals $(-2,-1)$, $(0,1)$ and $(1,2)$. Therefore, f has at least three real roots. On the other hand,

$$f''(x) = 60x^3 + 30x = 30x(2x^2+1)$$

has exactly one real root, so the mean value theorem (or its corollary Rolle's theorem, if you wish) implies that f has at most three real roots. Therefore, f has exactly three real roots.

b. If $g(x) = f(x) - x$, then $g'(x) = f'(x) - 1$ for all real numbers x, so the the intermediate value theorem and the mean value theorem apply to g on every closed interval of positive length.

If $f(a) = a$ or $f(b) = b$ then f has a fixed point. Otherwise $a < f(a) \leqslant b$ and $a \leqslant f(b) < b$, so $g(a) > 0$ and $g(b) < 0$, and the intermediate value theorem implies that there is a real number p such that $a < p < b$ and $g(p) = 0$, *i.e.*, $f(p) = p$. This proves that f has a fixed point p. For $q \neq p$ the mean value theorem implies that there is a real number ξ between p and q such that

$$g(q) - g(p) = g'(\xi)(q-p), \qquad \text{or equivalently} \qquad f(q) - q = \big(f'(\xi)-1\big)(q-p).$$

Since $q \neq p$ and $f'(\xi) \neq 1$, it follows that $f(q) \neq q$. Therefore, p is the only fixed point of f.

c. As c is the x intercept of the line tangent to the graph of f at the point $(b, f(b))$,

$$\frac{f(b)}{b-c} = f'(b), \qquad \text{or} \qquad c = b - \frac{f(b)}{f'(b)}, \qquad \text{and so} \qquad f'(b)(c-a) = f'(b)(b-a) - f(b).$$

By the mean value theorem there is a real number ξ such that $a < \xi < b$ and $f(b) = f'(\xi)(b-a)$. Likewise, there is a real number η such that $\xi < \eta < b$ and $f'(b) - f'(\xi) = f''(\eta)(b-\xi)$. Thus,

$$f'(b)(c-a) = \big(f'(b) - f'(\xi)\big)(b-a) = f''(\eta)(b-\xi)(b-a).$$

Now $|f'(b)| \geqslant 1$, $f''(\eta) \leqslant 1$ (because $a < \xi < \eta < b$) and $|b-\xi| < (b-a)$. Therefore,

$$|a-c| \leqslant |f'(b)(c-a)| = |f''(\eta)(b-\xi)(b-a)| < (a-b)^2.$$

Solution to Question 4. — The domain of $y = (\log x)^2/x$ is $(0,\infty)$, and its derivative,

$$\frac{dy}{dx} = \frac{2\log x}{x^2} - \frac{(\log x)^2}{x^2} = \frac{(\log x)(2-\log x)}{x^2},$$

is negative if $0 < x < 1$ or else $x > e^2$, and positive if $1 < x < e^2$. Therefore, y is decreasing on the intervals $(0,1]$ and $[e^2,\infty)$, and increasing on the interval $[1,e^2]$. The minimum value of y is 0, which occurs where $x = 1$; this is a global minimum value since $y \geqslant 0$ for all x in its domain. Also, $4e^{-2}$ is the local maximum value of y, which occurs where $x = e^2$; y has no global maximum value because $y \to \infty$ as $x \to 0^+$.

Solution to Question 5. — The domain of f is $\mathbb{R}\setminus\{0\}$. and $f(x)\to\pm\infty$ as $x\to 0^{\pm}$, so the y-axis is the vertical asymptote of the graph of f. (and f has no global extrema). As $x\to\pm\infty$, $f(x)\to 2$, so the line defined by $y=2$ is the horizontal asymptote of the graph of f. The axis intercepts of the graph of f are $(-\frac{3}{2},0)$ and $(3,0)$. The derivative

$$f'(x)=\frac{9}{x^2}-\frac{81}{x^4}=\frac{9(x^2-9)}{x^4}$$

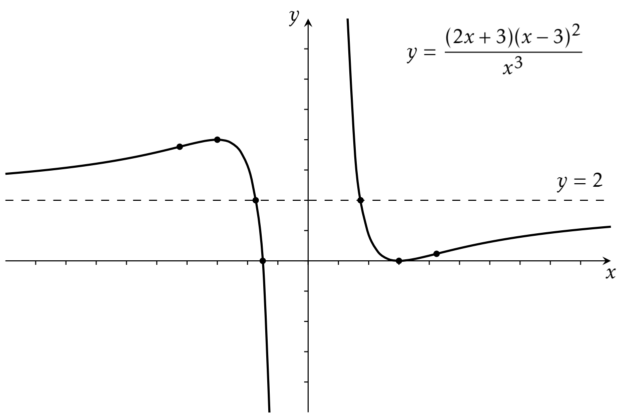
is positive if $|x|>3$ and negative if if $|x|<3$ and $x\neq 0$, so f is increasing on the intervals $(-\infty,-3)$ and $(3,\infty)$, and decreasing on the intervals $(-3,0)$ and on $(0,3)$, with a local maximum at $(-3,4)$ and a local minimum at $(3,0)$. The second derivative,

$$f''(x)=9\left\{-\frac{2}{x^3}+\frac{36}{x^5}\right\}=\frac{18(18-x^2)}{x^5}$$

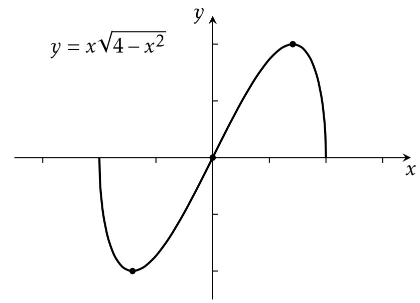
is positive if $x<-3\sqrt{2}$ or $0<x<3\sqrt{2}$ and negative if $-3\sqrt{2}<x<0$ or $x>3\sqrt{2}$, so the graph of f is convex on the intervals $(-\infty,-3\sqrt{2})$ and on $(0,3\sqrt{2})$, and concave down on the intervals $(-3\sqrt{2},0)$ and on $(3\sqrt{2},\infty)$, with points of inflection at $(\pm 3\sqrt{2}, 2\mp\frac{5}{4}\sqrt{2})$. The graph of f meets its horizontal asymptote where

$$0=-\frac{9}{x}+\frac{27}{x^3}=\frac{9(3-x^2)}{x^3},$$

i.e., at $(\pm\sqrt{3},2)$. In the following sketch unit lengths are marked along the coordinate axes, the horizontal asymptote drawn as a dashed line and the points of interest are emphasized.



Solution to Question 6. — The domain is $[-2,2]$, on which y is a continuous function of x, so the curve has no asymptotes. Since $y=0$ if, and only if, $x=0,\pm 2$, the intercepts of the curve are $(\pm 2,0)$ and the origin. Next, $\frac{dy}{dx}<0$ if $-2<x<-\sqrt{2}$ or $\sqrt{2}<x<2$ and $\frac{dy}{dx}>0$ if $-\sqrt{2}<x<\sqrt{2}$, so y is increasing on the interval $[-\sqrt{2},\sqrt{2}]$, and is decreasing on the intervals $[-2,-\sqrt{2}]$ and $[\sqrt{2},2]$. The local (and global) extreme values of f occur at the points $(\pm\sqrt{2},\pm 2)$. Since $y/(x+2)\to-\infty$ as $x\to -2^{+}$ and as $x\to 2^{-}$, the graph of f has one-sided vertical tangents at $(\pm 2,0)$. Finally, $\frac{d^2y}{dx^2}>0$ if $-2<x<0$ and $\frac{d^2y}{dx^2}<0$ if $0<x<2$, so the curve is concave up on the interval $[-2,0]$, and concave down on the interval $[0,2]$, with the origin as its sole point of inflection. The curve is sketched below, with unit lengths marked along the coordinate axes, and the points of interest emphasized.



Solution to Question 7. — If x is the radius of the cylinder, and y is half its height then $x^2+y^2=r^2$ (the latter by Pythagoras' formula). Hence, the volume of the cylinder is

$$V=\pi x^2(2y)=2\pi(r^2-y^2)y=2\pi(r^2y-y^3),$$

and the surface area of the cylinder is

$$A=2\pi x^2+2\pi x(2y)=2\pi(x^2+2xy).$$

To find the largest possible volume of the cylinder, observe that

$$\frac{dV}{dy}=2\pi(r^2-3y^2)\qquad\text{and}\qquad\frac{d^2V}{dy^2}=-12\pi y<0,$$

since $0<y<r$, so by the second derivative test for global extreme values, the largest volume occurs where $r^2=3y^2$, or $y=\frac{1}{3}\sqrt{3}r$ and is $2\pi\left(\frac{2}{3}r^2\right)\left(\frac{1}{3}r\sqrt{3}\right)=\frac{4}{9}\sqrt{3}\pi r^3$.

Next, observe that $A=0$ if $x=0$ or $x=r$, so the largest possible surface area occurs at a critical number in $(0,r)$. Now

$$\frac{dy}{dx}=-\frac{x}{y},\qquad\text{and so}\qquad\frac{dA}{dx}=4\pi\left(x+y+x\frac{dy}{dx}\right)=4\pi\frac{xy+y^2-x^2}{y}$$

which vanishes where $x^2-y^2=xy$. Since $x^2+y^2=r^2$, it follows that

$$r^4-(xy)^2=(x^2+y^2)^2-(x^2-y^2)^2=4(xy)^2,\qquad\text{or}\qquad r^4=5(xy)^2.$$

Therefore,

$$4xy=\tfrac{4}{5}\sqrt{5}\,r^2\qquad\text{and}\qquad 2x^2=r^2+xy=\left(1+\tfrac{1}{5}\sqrt{5}\right)r^2,$$

and the largest possible surface area of the cylinder is $\pi(2x^2+4xy)=\pi(1+\sqrt{5})r^2$.

Solution to Question 8. — a. Since $(2^x-3^x)^2/6^x=\left(\frac{2}{3}\right)^x+\left(\frac{3}{2}\right)^x-2$, the integral is equal to

$$\int_{-1}^{1}\left\{\left(\tfrac{2}{3}\right)^x+\left(\tfrac{3}{2}\right)^x-2\right\}dx=\left\{\frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)}+\frac{\left(\frac{3}{2}\right)^x}{\log\left(\frac{3}{2}\right)}-2x\right\}\Bigg|_{-1}^{1}=\frac{5}{3\log\left(\frac{3}{2}\right)}-4.$$

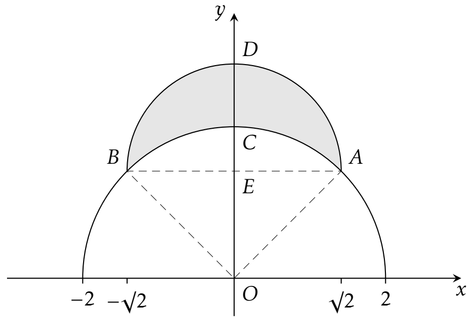
b. As $(y-1)^3/(y^3-y^2)=(y-1)^2/y^2=(y^2-2y+1)/y^2=1-2/y+1/y^2$, the integral is equal to

$$\int\left(1-\frac{2}{y}+\frac{1}{y^2}\right)dy=y-2\log|y|-\frac{1}{y}+C.$$

c. Since $(\sin\vartheta)/(1+\sin\vartheta)=(\sin\vartheta)(1-\sin\vartheta)/\cos^2(\vartheta)=\sec(\vartheta)\tan(\vartheta)-\sec^2(\vartheta)+1$, the integral is equal to

$$\int\left(\sec(\vartheta)\tan(\vartheta)-\sec^2(\vartheta)+1\right)d\vartheta=\sec(\vartheta)-\tan(\vartheta)+\vartheta+C.$$

d. The integral is equal to the area of the region below the curve defined by $y = \sqrt{2} + \sqrt{2 - x^2}$ and above the curve defined by $y = \sqrt{4 - x^2}$, over the interval $[-\sqrt{2}, \sqrt{2}]$ on the x-axis. Each curve is a circular arc, as in the figure below, where the region in question is shaded. In the figure, $|OA| = |OB| = |OC| = 2$ and $|EA| = |EB| = |ED| = \sqrt{2}$.



The area of the quarter circle $OACB$ is equal to the area of the semicircle $EADB$. Subtracting the area of the circular segment $EACB$ from the area of the quarter circle leaves the area of the triangle OAB, or 2. Subtracting the area of the circular segment $EACB$ from the area of the semicircle $EADB$ leaves the area of the shaded region. Therefore,

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left\{ \sqrt{2} + \sqrt{2 - x^2} - \sqrt{4 - x^2} \right\} dx = 2.$$

Solution to Question 9. — a. If the interval $[-1, 3]$ is divided into k subintervals of equal length, then the length of each subinterval is $\frac{4}{k}$ and the endpoints of the subintervals are $x_j = -1 + \frac{4}{k} j$, for $j = 0, 1, 2, \ldots, k$. If $f(x) = x^3 - 4x + 2$, then

$$\begin{aligned} f(x_j) &= \left(-1 + \frac{4}{k} j\right)^3 - 4\left(-1 + \frac{4}{k} j\right) + 2 = -1 + \frac{12}{k} j - \frac{48}{k^2} j^2 + \frac{64}{k^3} j^3 + 4 - \frac{16}{k} j + 2 \\ &= 5 - \frac{4}{k} j - \frac{48}{k^2} j^2 + \frac{64}{k^3} j^3, \end{aligned}$$

and the corresponding right endpoint Riemann sum is given by

$$\begin{aligned} \mathscr{R}_n &= \frac{4}{k} \sum_{j=1}^{k} \left\{ 5 - \frac{4}{k} j - \frac{48}{k^2} j^2 + \frac{64}{k^3} j^3 \right\} = 20 - \frac{16}{k^2} \sum_{j=1}^{k} j - \frac{192}{k^3} \sum_{j=1}^{k} j^2 + \frac{256}{k^4} \sum_{j=1}^{k} j^3 \\ &= 20 - \frac{16}{k^2} \cdot \frac{1}{2} k(k+1) - \frac{192}{k^3} \cdot \frac{1}{6} k(k+1)(2k+1) + \frac{256}{k^4} \cdot \frac{1}{4} k^2 (k+1)^2 \\ &= 20 - 8\left(1 + \frac{1}{k}\right) - 32\left(1 + \frac{1}{k}\right)\left(2 + \frac{1}{k}\right) + 64\left(1 + \frac{1}{k}\right)^2. \end{aligned}$$

Therefore,

$$\int_{-1}^{3} (x^3 - 4x + 2)\, dx = \lim_{n \to \infty} \mathscr{R}_n = 20 - 8 - 64 + 64 = 12.$$

If the interval $[\,\frac{1}{3}\pi, \frac{7}{6}\pi\,]$ is divided into n subintervals of equal length, then the length of each subinterval is $\frac{5}{6n}\pi$, the endpoints of the subintervals are $\frac{1}{3}\pi + \frac{5}{6n}\pi i$, and the values of $g(x) = \cos(3x)$ at these endpoints are $\cos\left(3(\frac{1}{3}\pi + \frac{5}{6n}\pi i)\right) = \cos\left(\pi + \frac{5}{2n}\pi i\right)$, for $i = 0, 1, 2, \ldots, n$. The Riemann sum obtained by evaluating the integrand at the right endpoints of the subintervals is

$$\mathscr{R}_n = \frac{5\pi}{6n} \sum_{i=1}^{n} \cos\left(\pi + \frac{5\pi}{2n} i\right) = \frac{5\pi}{6n} \cdot \frac{\sin\left(\pi + (n + \frac{1}{2}) \frac{5\pi}{2n}\right) - \sin\left(\pi + \frac{5\pi}{4n}\right)}{2 \sin\left(\frac{5\pi}{4n}\right)},$$

using the identity $2\sin\left(\frac{1}{2}\beta\right)\cos(\alpha + i\beta) = \sin\left(\alpha + (i + \frac{1}{2})\beta\right) - \sin\left(\alpha + (i - \frac{1}{2})\beta\right)$, which follows from the sum and difference identities of the sine function. Therefore,

$$\begin{aligned} \int_{\frac{1}{3}\pi}^{\frac{7}{6}\pi} \cos(3x)\, dx &= \lim_{n \to \infty} \mathscr{R}_n = \frac{5\pi}{12} \left(\sin\left(\pi + \frac{5}{2}\pi\right) - \sin(\pi) \right) \cdot \lim_{n \to \infty} \frac{1}{n \sin\left(\frac{5\pi}{4n}\right)} \\ &= -\frac{5\pi}{12} \cdot \lim_{x \to 0} \frac{4x}{5\pi \sin(x)} = -\frac{5\pi}{12} \cdot \frac{4}{5\pi} = -\frac{1}{3}, \end{aligned}$$

where $x = \frac{5\pi}{4n}$.

b. i. If the interval $[0, 1]$ is divided into n subintervals of equal length, then the width of each subinterval is $\frac{1}{n}$ and the endpoints of the subintervals are $\frac{k}{n}$, for $k = 0, 1, 2, \ldots, n$. The corresponding limit of regular right endpoint Riemann sums of $\ln(3 - x)$ gives the definite integral

$$\int_0^1 \ln(3 - x)\, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln(3 - k/n).$$

ii. If the interval $[-5, -4]$ is divided into n subintervals of equal length, then the width of each subinterval is $\frac{1}{n}$ and the endpoints of the subintervals are $-5 + \frac{k}{n}$. The corresponding limit of regular right endpoint Riemann sums of $\ln(-2 - x)$ gives the definite integral

$$\int_{-5}^{-4} \ln(-2 - x)\, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(-2 - \left(-5 + \frac{k}{n}\right)\right) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln(3 - k/n).$$

iii. If the interval $[1, 2]$ is divided into n subintervals of equal length, then the width of each subinterval is $\frac{1}{n}$ and the endpoints of the subintervals are $1 + \frac{k}{n}$. The corresponding limit of regular right endpoint Riemann sums of $\ln(4 - x)$ gives the definite integral

$$\int_1^2 \ln(4 - x)\, dx = \lim_{k \to \infty} \sum_{k=1}^{n} \ln\left(4 - \left(1 + \frac{k}{n}\right)\right) = \lim_{k \to \infty} \sum_{k=1}^{n} \ln(3 - x).$$

Solution to Question 10. — a. Let y denote the right side of the integral formula. Logarithmic differentiation gives

$$\begin{aligned} \frac{dy}{dx} &= y \frac{d}{dx}\left\{ \log|y| \right\} = y \cdot \left\{ \frac{8x}{4x^2 - 1} + \frac{2x}{2x^2 + 1} - \frac{3}{x} \right\} = y \cdot \frac{16x^4 + 8x^2 + 8x^4 - 2x^2 - 24x^4 - 6x^2 + 3}{(4x^2 - 1)(2x^2 + 1)x} \\ &= \frac{(4x^2 - 1)\sqrt{2x^2 + 1}}{3x^3} \cdot \frac{3}{(4x^2 - 1)(2x^2 + 1)x} = \frac{1}{x^4 \sqrt{2x^2 + 1}}, \end{aligned}$$

as required. The second fundamental theorem of calculus then gives

$$\int_1^2 \frac{dx}{x^4 \sqrt{2x^2 + 1}} = \left. \frac{(4x^2 - 1)\sqrt{2x^2 + 1}}{3x^3} \right|_1^2 = \frac{15}{8} - \sqrt{3}.$$

b. Differentiating, using the first fundamental theorem of calculus and the chain rule, gives

$$f(x) = -\frac{\log(x)}{x^2} + \frac{1}{x^2} - \frac{f(x)}{x^2} \cdot 2x, \qquad \text{or equivalently} \qquad \frac{x+2}{x} f(x) = \frac{1-\log(x)}{x^2}.$$

Therefore,

$$f(x) = \frac{\log(e/x)}{x(x+2)}.$$