FIRST BONUS QUESTION

To submit a solution, print this document, print your full name on the top right corner of the page and write your solution in space below the question. Solutions must be written in good mathematical English; use complete sentences, and correct notation and terminology.

DEADLINE: Friday 9 November at 8h15. [The solution will be posted at 8h25.]

Question. — Let y be a positive real number which is not equal to 1. Given $\varepsilon > 0$, find a positive integer n_o such that $|\sqrt[n]{y} - 1| < \varepsilon$, provided n is a positive integer and $n > n_o$. Express n_o (explicitly and simply) in terms of y and ε, and justify all assertions (especially all inequalities) carefully. You may find it advantageous to consider separately the cases in which $0 < y < 1$ and $y > 1$.

Solution. — For $y > 0$ the factorization of a difference of powers yields

$$|\sqrt[n]{y} - 1| = \frac{|y-1|}{1 + \sqrt[n]{y} + \sqrt[n]{y}^2 + \cdots + \sqrt[n]{y}^{n-1}}. \tag{$*$}$$

If $0 < y < 1$ and $j = 1, 2, \ldots, n-1$, then $0 < y^n < y^j$, or $0 < y < \sqrt[n]{y}^j$, and hence

$$1 + \sqrt[n]{y} + \sqrt[n]{y}^2 + \cdots + \sqrt[n]{y}^{n-1} > 1 + (n-1)y = 1 - y + ny. \tag{$\dagger$}$$

If $y > 1$ and $j = 1, 2, \ldots, n-1$, then $\sqrt[n]{y}^j > 1$, so

$$1 + \sqrt[n]{y} + \sqrt[n]{y}^2 + \cdots + \sqrt[n]{y}^{n-1} > n. \tag{$\ddagger$}$$

Given $\varepsilon > 0$, let n_o be the smallest positive integer which is larger than

$$\begin{cases} \dfrac{1-y}{\varepsilon y} & \text{if } 0 < y < 1, \text{ and} \\ \dfrac{y-1}{\varepsilon} & \text{if } y > 1. \end{cases}$$

If $n > n_o$ and $0 < y < 1$, then $(*)$ and $(\dagger)$ imply that

$$|\sqrt[n]{y} - 1| < \frac{1-y}{1-y+ny} = \frac{1}{1 + \dfrac{ny}{1-y}} < \frac{1}{1 + \dfrac{1}{\varepsilon}} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon, \qquad \text{since} \qquad n > n_o > \frac{1-y}{\varepsilon y}.$$

If $n > n_o$ and $y > 1$, then $(*)$ and $(\ddagger)$ imply that

$$|\sqrt[n]{y} - 1| < \frac{y-1}{n} < \frac{y-1}{\dfrac{y-1}{\varepsilon}} < \varepsilon, \qquad \text{since} \qquad n > n_o > \frac{y-1}{\varepsilon}.$$

So in any case, $|\sqrt[n]{y} - 1| < \varepsilon$ provided $n > n_o$. □
