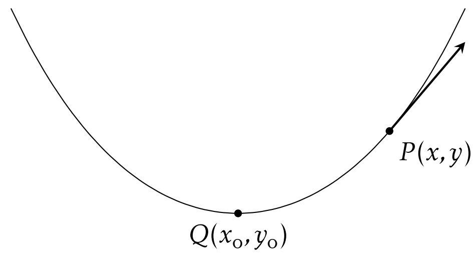
SOME QUESTIONS FOR REVIEW

Question 1. — A flexible cable which is suspended from two points of equal height forms a curve as illustrated below (at least approximately).



If τ denotes the tension in the cable at the point P, and h, v denote, respectively, the horizontal and vertical components of τ, then $h = \tau\cos\varphi$ and $v = \tau\sin\varphi$, where φ is the angle of inclination of the tangent to the curve at P. Assuming that the cable has uniform density (and choosing convenient coordinates and units of tension), determine an equation of the curve by considering v/h, solving a differential equation for dy/dx, and finally integrating to obtain y as a function of x.

Question 2. — For each series below,

i. give a formula for the sum of the first k terms of the series, and
ii. use this formula to compute the sum of the series, or else to show that the series is divergent.

a. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2k+3)\cdots(2k+2019)}$ b. $\displaystyle\sum_{k=1}^{\infty}\frac{3\cdot 2^{4k}-4\cdot 3^{3k}}{5^{2k}}$ c. $\displaystyle\sum_{k=2}^{\infty}\log\sqrt[3]{\frac{12k^2-k-20}{12k^2-k-1}}$

Question 3. — Investigate completely the convergence of the series

$$\sum_{k=1}^{\infty}(-1)^k(1-\cos(k^a)),$$

in which a is a real number. That is, determine the values of a for which the given series is: i. absolutely convergent; ii. conditionally convergent; iii. divergent.

Question 4. — Investigate completely the convergence of each of each series below (include a determination of absolute versus conditional convergence where appropriate).

a. $\displaystyle\sum_{k=1}^{\infty}\frac{(-k)^k k!}{(2k)!}$ b. $\displaystyle\sum_{n=1}^{\infty}(\sqrt[n]{\pi}-1)$ c. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{(25k+3)(\log k)^2}{(2k-7)^3}$ d. $\displaystyle\sum_{n=2}^{\infty}\frac{\sec^2(n)}{n\sqrt{\log(n)}}$

e. $\displaystyle\sum_{n=1}^{\infty}(-1)^n n\sin(1/n^2)$ f. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n\sqrt[n]{5}}{n+2}\right)^{n^2}$ g. $\displaystyle\sum_{j=1}^{\infty}\frac{(-1)^j j^5}{e^{\sqrt[3]{j}}}$ h. $\displaystyle\sum_{n=1}^{\infty}\cos(\pi n)\operatorname{arccsc}(\sqrt{n})$

Question 5. — Expand each function into a power series about the given centre:

a. $f(x) = x^2 e^{-2x}$ about 1; b. $f(x) = \dfrac{(x-3)^2}{2x^2-x-1}$ about 3.

SOLUTIONS TO SOME QUESTIONS FOR REVIEW

Solution to question 1. — If the cable has uniform density then v is proportional to the length of the part of the cable between Q (the lowest point of the curve) and P; *i.e.*, there is a positive real number α such that

$$\alpha v = \int_0^x \sqrt{1 + \left(\frac{dy}{d\xi}\right)^2}\, d\xi.$$

Also, h is constant—it is equal to the horizontal component of the tension at Q—so choose units of tension so that $h = 1$. Then

$$\frac{dy}{dx} = \tan\varphi = \frac{v}{h} = \frac{1}{\alpha}\int_0^x \sqrt{1 + \left(\frac{dy}{d\xi}\right)^2}\, d\xi,$$

and so

$$\frac{d^2y}{dx^2} = \frac{1}{\alpha}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \qquad \text{or} \qquad \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\alpha}.$$

Choosing coordinates so that Q is $(0, \alpha)$ and integrating yields

$$\log\left\{\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right\} = \frac{x}{\alpha}, \qquad \text{since} \qquad \frac{dy}{dx} = 0 \quad \text{at} \quad Q, \quad \text{where} \quad x = 0.$$

Applying the exponential function and rearranging then gives

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(e^{x/\alpha} - \frac{dy}{dx}\right)^2, \qquad i.e., \qquad 2e^{x/\alpha}\frac{dy}{dx} = e^{2x/\alpha} - 1, \qquad \text{or} \qquad \frac{dy}{dx} = \tfrac{1}{2}\left(e^{x/\alpha} - e^{-x/\alpha}\right).$$

Integrating again gives

$$y = \tfrac{1}{2}\alpha\left(e^{x/\alpha} - e^{-x/\alpha}\right),$$

since $y = \alpha$ at Q, where $x = 0$.

Solution to question 2. — a. Notice that (by combining the fractions on the left)

$$\frac{1}{(2k+1)(2k+3)\cdots(2k+2017)} - \frac{1}{(2k+3)(2k+5)\cdots(2k+2019)} = \frac{2018}{(2k+1)(2k+3)\cdots(2k+2019)},$$

so if a_k denotes the general term of a given series and

$$A_k = \frac{1}{2018} \cdot \frac{1}{(2k+1)(2k+3)\cdots(2k+2017)},$$

then $a_k = A_k - A_{k+1}$ for $k \geqslant 0$, and the sum of the first k terms of the series is

$$a_0 + a_1 + \cdots + a_{k-1} = A_0 - A_k = \frac{1}{2018} \cdot \frac{1}{1 \cdot 3 \cdots 2017} - \frac{1}{2018} \cdot \frac{1}{(2k+1)(2k+3)\cdots(2k+2017)}.$$

Since $\lim A_k = 0$, the sum of the series is equal to

$$\lim(a_0 + a_1 + \cdots + a_{k-1}) = \frac{1}{2018} \cdot \frac{1}{1 \cdot 3 \cdots 2017} = \frac{2^{1008} \cdot 1008!}{2018!},$$

where the last expression is obtained by multiplying and dividing by $2 \cdot 4 \cdots 2016 = 2^{1008} \cdot 1008!$.

b. The general term of the series is $3\left(\frac{16}{25}\right)^k - 4\left(\frac{27}{25}\right)^k$, so the sum of the first k terms of the series is

$$3 \cdot \frac{16}{25} \cdot \frac{1 - \left(\frac{16}{25}\right)^k}{1 - \frac{16}{25}} - 4 \cdot \frac{27}{25} \cdot \frac{1 - \left(\frac{27}{25}\right)^k}{1 - \frac{27}{25}} = \tfrac{16}{3}\left(1 - \left(\tfrac{16}{25}\right)^k\right) + 54\left(1 - \left(\tfrac{27}{25}\right)^k\right),$$

using the factorization $1 - x^k = (1-x)(1 + x + \cdots + x^{k-1})$ of a difference of powers. Since $\left(\frac{16}{25}\right)^k \to 0$ and $\left(\frac{27}{25}\right)^k \to \infty$, it follows that the series in question diverges to $-\infty$.

c. The general term of the series is

$$\frac{1}{3}\log\left\{\frac{(3k-4)(4k+5)}{(4k+1)(3k-1)}\right\} = \frac{1}{3}\log\left(\frac{3k-4}{4k+1}\right) - \frac{1}{3}\log\left(\frac{3k-1}{4k+5}\right),$$

so if a_k denotes the general term of the series and $A_k = \frac{1}{3}\log\left(\frac{3k-4}{4k+1}\right)$, then $a_k = A_k - A_{k+1}$ for $k \geqslant 2$ and the sum of the first k terms of the series is

$$a_2 + a_3 + \cdots + a_{k+1} = A_2 - A_{k+2} = \tfrac{1}{3}\log\left(\tfrac{2}{9}\right) - \frac{1}{3}\log\left(\frac{3k+2}{4k+9}\right) = \frac{1}{3}\log\left\{\frac{2(4k+9)}{9(3k+2)}\right\}.$$

The sum of the series is $\lim(a_2 + a_3 + \cdots + a_{k+1}) = \frac{1}{3}\log\left(\frac{8}{27}\right) = \log\left(\frac{2}{3}\right)$, where the limit is computed by inspecting dominant terms.

Solution to question 3. — Let $a_k = 1 - \cos(k^a)$; notice that $a_k \geqslant 0$ for $k \geqslant 1$. If $a = 0$ then $a_k = 0$ for $k \geqslant 1$, so $\sum(-1)^k a_k$ is absolutely convergent. If $a > 0$ then $\{a_k\}$ does not converge to 0, so the vanishing condition implies that $\sum(-1)^k a_k$ is divergent. If $a < 0$ then $k^a \to 0$, so $1 - \cos(k^a) \to 0$. Since $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^5 + \&c.$, with $x = k^a$, it follows that

$$\lim \frac{1-\cos(k^a)}{k^{2a}} = \lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \tfrac{1}{2},$$

which is positive and finite. Since $\sum k^{2a}$ is a p-series ($p = -2a$) which converges if, and only if, $a < -\frac{1}{2}$, the limit comparison test implies that $\sum a_k$ is convergent if $a < -\frac{1}{2}$ and divergent if $-\frac{1}{2} \leqslant a < 0$. On the other hand, if $a < 0$ and $k \geqslant 1$ then $0 < (k+1)^a < k^a < \frac{1}{2}\pi$, and thus

$$0 < 1 - \cos\big((k+1)^a\big) < 1 - \cos(k^a).$$

Since $1 - \cos(k^a) \to 0$ if $a < 0$, the Leibniz test implies that $\sum(-1)^k a_k$ is convergent if $a < 0$.

Hence, the series $\sum(-1)^k a_k$ is absolutely convergent if $a < -\frac{1}{2}$ or $a = 0$, conditionally convergent if $-\frac{1}{2} \leqslant a < 0$, and divergent if $a > 0$.

Solution to question 4. — a. If $a_k = k^k k!/(2k)!$ then $a_k > 0$ if $k \geqslant 1$, and the general term of the series is $(-1)^k a_k$. Now

$$\lim \frac{a_{k+1}}{a_k} = \lim \frac{(k+1)^{k+1}\,(k+1)!(2k)!}{k^k\,k!(2k+2)!} = \lim\left\{\left(1+\frac{1}{k}\right)^k \cdot \frac{k+1}{2(2k+1)}\right\} = \tfrac{1}{4}e < 1,$$

so the ratio test implies that $\sum a_k$ is convergent. Therefore, $\sum(-1)^k a_k$ is absolutely convergent.

b. Notice that $\pi^{1/n} - 1 > 0$ for $n \geqslant 1$. If $x = \pi^{1/n}$ then $n = \log(\pi)/\log(x)$, and so

$$\lim \frac{\pi^{1/n}-1}{1/n} = \log(\pi) \cdot \lim_{x\to 1} \frac{x-1}{\log(x)} = \log(\pi),$$

which is positive. Thus, the limit comparison test implies that the series $\sum(\sqrt[n]{\pi} - 1)$ diverges with the harmonic series (*i.e.*, the p-series for which $p = 1$).

c. If

$$a_k = \frac{(25k+3)(\log k)^2}{(2k-7)^3} \qquad \text{and} \qquad b_k = \frac{1}{k^{3/2}},$$

then $a_k > 0$ and $b_k > 0$ if $k \geqslant 4$, and

$$\lim \frac{a_k}{b_k} = \lim \frac{k^{3/2}(25k+3)(\log k)^2}{(2k-7)^3} = \lim\left\{\frac{25+3/k}{(2-7/k)^3} \cdot \frac{(\log k)^2}{k^{1/2}}\right\} = 0,$$

which is finite, so the limit comparison test implies that the series $\sum a_k$ converges with the p-series $\sum b_k$ ($p = \frac{3}{2} > 1$). Therefore, the series $\sum(-1)^k a_k$ is absolutely convergent.

d. If $n \geqslant 2$ then $\sec^2(n) > 1$, and so

$$\frac{\sec^2(n)}{n\sqrt{\log(n)}} > \frac{1}{n\sqrt{\log(n)}} > 0,$$

so the comparison test implies that the series in question diverges with the logarithmic p-series $\sum\left(n\sqrt{\log(n)}\right)^{-1}$ ($p = \frac{1}{2} \geqslant 1$).

e. If $a_n = n\sin(1/n^2)$ and $b_n = 1/n$, then $a_n > 0$ and $b_n > 0$ if $n \geqslant 1$, and

$$\lim \frac{a_n}{b_n} = \lim \frac{\sin(1/n)}{1/n} = 1,$$

which is positive, so the limit comparison test implies that the series $\sum a_k$ diverges with the harmonic series (*i.e.*, the p-series for which $p = 1$). On the other hand, if $x > 1$ then $\sin(1/x^2) < 1/x^2$ and $\cos(1/x^2) > \frac{1}{2}$, so

$$\frac{d}{dx}\Big\{x\sin(1/x^2)\Big\} = \sin(1/x^2) - \frac{2\cos(1/x^2)}{x^2} < \frac{1}{x^2} - \frac{1}{x^2} = 0,$$

which implies that $0 < (n+1)\sin\Big(1/(n+1)^2\Big) < n\sin(1/n^2)$ if $n \geqslant 1$. Also

$$\lim\Big\{n\sin(1/n^2)\Big\} = \lim\left\{\frac{1}{n} \cdot \frac{\sin(1/n^2)}{1/n^2}\right\} = 0 \cdot 1 = 0,$$

so the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum(-1)^n a_n$ is conditionally convergent.

f. If

$$a_n = \left(\frac{n\sqrt[n]{5}}{n+2}\right)^{n^2},$$

then $a_n \geqslant 0$ for $n \geqslant 1$ and

$$\lim \sqrt[n]{a_n} = \lim\left\{5\left(1 - \frac{2}{n+2}\right)^{\frac{n+2}{-2} \cdot \frac{-2n}{n+2}}\right\} = 5e^{-2} < 1,$$

so the root test implies that the series $\sum a_n$ is convergent.

g. If $a_j = j^5 e^{-j^{1/3}}$ and $b_j = 1/j^2$, then $a_j > 0$ and $b_j > 0$ for $j \geqslant 1$, and

$$\lim \frac{a_j}{b_j} = \lim \frac{j^7}{e^{j^{1/3}}} = 0,$$

which is finite, so the limit comparison test implies that the series $\sum a_j$ converges with the p-series $\sum b_j$ ($p = 2 > 1$). Therefore, the series $\sum(-1)^j a_j$ is absolutely convergent.

h. If $a_n = \operatorname{arccsc}(\sqrt{n})$ and $b_n = n^{-1/2}$, then $a_n > 0$ and $b_n > 0$ for $n \geqslant 1$. If $\vartheta = \operatorname{arccsc}(\sqrt{n})$, then $\sqrt{n} = 1/(\sin\vartheta)$, so

$$\lim \frac{a_n}{b_n} = \lim\Big\{\sqrt{n}\operatorname{arccsc}(\sqrt{n})\Big\} = \lim_{\vartheta\to 0} \frac{\vartheta}{\sin(\vartheta)} = 1,$$

which is positive, so the limit comparison test implies that the series $\sum a_n$ diverges with the p-series $\sum b_n$ ($p = \frac{1}{2} \leqslant 1$). On the other hand, $\operatorname{arccsc}(\sqrt{n}) > \operatorname{arccsc}(\sqrt{n+1}) > 0$ if $n \geqslant 1$ and $\lim \operatorname{arccsc}(\sqrt{n}) = 0$, so the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent. Therefore, the series $\sum(-1)^n a_n$ is conditionally convergent. Notice that $\cos(\pi n)$ is just another notation for $(-1)^n$.

Solution to question 5. — a. Revising the expression defining $f(x)$ and using the Maclaurin expansion of the exponential function gives

$$\begin{aligned}
x^2e^{-2x} &= e^{-2}\Big\{1 + 2(x-1) + (x-1)^2\Big\}e^{-2(x-1)} = e^{-2}\Big\{1 + 2(x-1) + (x-1)^2\Big\}\sum_{k=0}^{\infty} \frac{(-2)^n}{n!}(x-1)^n \\
&= e^{-2}\sum_{k=0}^{\infty} \frac{(-2)^n}{n!}\Big\{(x-1)^n + 2(x-1)^{n+1} + (x-1)^{n+2}\Big\} \\
&= e^{-2} + \sum_{k=2}^{\infty}\left\{\frac{(-2)^n}{n!} + \frac{2(-2)^{n-1}}{(n-1)!} + \frac{(-2)^{n-2}}{(n-2)!}\right\}(x-1)^n \\
&= \sum_{k=0}^{\infty} \frac{(-2)^n(n-1)(n-4)}{4e^2\, n!}(x-1)^n,
\end{aligned}$$

which is valid for all real values of x. (It so happens, in this case, that the simplified form of the general term incorporates the first two terms of the series.)

b. The resolution into partial fractions, elementary revision, and the formula for the sum of a geometric series gives

$$\begin{aligned}
\frac{1}{2x^2-x-1} = \frac{1}{(x-1)(2x+1)} = \frac{1/3}{x-1} - \frac{2/3}{2x+1} &= \frac{1}{6}\cdot\frac{1}{1+\frac{1}{2}(x-3)} - \frac{2}{21}\cdot\frac{1}{1+\frac{2}{7}(x-3)} \\
&= \sum_{k=0}^{\infty}\left\{\frac{1}{6}\cdot\frac{(-1)^k}{2^k} - \frac{2}{21}\cdot\frac{(-1)^k 2^k}{7^k}\right\}(x-3)^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(7^{k+1}-2^{2k+2})}{3\cdot 14^{k+1}}(x-3)^k,
\end{aligned}$$

which is valid if $1 < x < 5$ (via the interval of convergence of a geometric series). Therefore,

$$\frac{(x-3)^2}{2x^2-x-1} = \sum_{k=0}^{\infty}\frac{(-1)^k(7^{k+1}-2^{2k+2})}{3\cdot 14^{k+1}}(x-3)^{k+2} = \sum_{k=2}^{\infty}\frac{(-1)^k(7^{k-1}-2^{2k-2})}{3\cdot 14^{k-1}}(x-3)^k,$$

which is also valid if $1 < x < 5$.