Exercise 1. — For each integral below, find a change of variables which transforms the integral into the integral of a rational function of y. To earn full credit:

- express y as a function of x;
- express x as a rational function of y;
- express dx as a rational function of y times dy.

You may use intermediate variables to arrive at the complete rationalization; however, to earn credit you must complete the tasks itemized above.

a. $\displaystyle\int \frac{5x^2 - x + 3 - 2\sqrt[3]{(2x+1)^2(x-2)^4}}{x^2}\,dx$

b. $\displaystyle\int \frac{(x^2 - x + 1)\sqrt[5]{(x-3)^4}}{\sqrt[5]{(x-1)^9}}\,dx$

Exercise 2. — For each integral below, find a change of variables which transforms the integral into the integral of a rational function of y. To earn full credit,

- express y as a function of x;
- express $\sin(x)$ and $\cos(x)$ as rational functions of y.
- express dx as a rational function of y times dy.

You may use intermediate variables to arrive at the complete rationalization; however, to earn credit you must complete the tasks itemized above.

a. $\displaystyle\int \frac{2\sin(x) - 3\cos(x) + 5}{\sin^3(x) - 2\cos(x) + 2}\,dx$

b. $\displaystyle\int \frac{2\tan(x) - 3}{1 - \sin(x) + 4\sec(x)}\,dx$

Exercise 3. — For each integral below, find a change of variables which transforms the integral into the integral of a rational function of y. To earn full credit:

- express y as a function of x;
- express the integral as the integral of a rational function of y.

You may use intermediate variables to arrive at the complete rationalization; however, to earn credit you must complete the tasks itemized above.

a. $\displaystyle\int \frac{\sqrt[3]{x^2}}{\sqrt[7]{(5 + 2x\sqrt[6]{x})^3}}\,dx$

b. $\displaystyle\int \frac{\sqrt[3]{(4 + 3\sqrt[3]{x^2})^5}}{\sqrt[3]{x^7}}\,dx$

Exercise 4. — Express the improper integral

$$\int_{-\infty}^{\frac{3}{2}} \frac{\log(x^2)}{\sqrt[3]{(x^2 - x - 2)(x^2 + x - 6)^2}}\,dx$$

as a sum of limits of definite integrals. In doing so you may write f for the integrand (including the differential).

Solution to exercise 1. — a. Observe that

$$\sqrt[3]{(2x+1)^2(x-2)^4} = (2x+1)(x-2)\sqrt[3]{\frac{x-2}{2x+1}}; \qquad \text{and let} \qquad y = \sqrt[3]{\frac{x-2}{2x+1}}.$$

Then $y^3 = \dfrac{x-2}{2x+1}$, and solving for x and differentiating gives

$$x = -\frac{y^3+2}{2y^3-1} \qquad \text{and} \qquad dx = \frac{15y^2}{(2y^3-1)^2}\,dy.$$

From the two displays it is plain that the integral in question is the integral of a rational function of y.

b. Observe that

$$\frac{\sqrt[5]{(x-3)^4}}{\sqrt[5]{(x-1)^9}} = \frac{x-3}{(x-1)^2}\sqrt[5]{\frac{x-1}{x-3}}; \qquad \text{and let} \qquad y = \sqrt[5]{\frac{x-1}{x-3}}.$$

Then $y^5 = \dfrac{x-1}{x-3}$, and solving for x and differentiating gives

$$x = \frac{3y^5-1}{y^5-1} \qquad \text{and} \qquad dx = -\frac{10y^4}{(y^5-1)^2}\,dy.$$

From the two displays it is plain that the integral in question is the integral of a rational function of y.

Solution to exercise 2. — If $-\pi < x < \pi$ and $y = \tan\left(\tfrac{1}{2}x\right)$, then

$$\sin\left(\tfrac{1}{2}x\right) = \frac{y}{\sqrt{1+y^2}}, \qquad \cos\left(\tfrac{1}{2}x\right) = \frac{1}{\sqrt{1+y^2}},$$

so

$$\sin(x) = 2\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{1}{2}x\right) = \frac{2y}{1+y^2} \qquad \text{and} \qquad \cos(x) = \cos^2(x) - \sin^2(x) = \frac{1-y^2}{1+y^2}.$$

Also, $x = 2\arctan(y)$, so $dx = \dfrac{2}{1+y^2}\,dy$. This change of variables applies to the integrals in part a and part b.

Solution to exercise 3. — a. Let $y = \dfrac{\sqrt[7]{5+2x\sqrt[6]{x}}}{\sqrt[6]{x}}$; then $y^7 = 5x^{-7/6}+2$ and $7y^6\,dy = -\frac{35}{6}x^{-13/6}\,dx$, or $-\frac{6}{5}y^6\,dy = \dfrac{dx}{x^{13/6}}$. Also, $x^{7/3} = \left(\dfrac{5}{y^7-2}\right)^2 = \dfrac{25}{(y^7-2)^2}$, and so

$$\int \frac{\sqrt[3]{x^2}}{\sqrt[7]{(5+2x\sqrt[6]{x})^3}}\,dx = \int x^{7/3}\cdot\left(\frac{x^{1/6}}{(5+2x^{7/6})^{1/7}}\right)^3\cdot\frac{dx}{x^{13/6}} = \int \frac{25}{(y^7-2)^2}\cdot\frac{1}{y^3}\cdot\left(-\tfrac{6}{5}y^6\,dy\right) = \int \frac{-30y^3}{(y^7-2)^2}\,dy.$$

b. Let $y = \sqrt[3]{4+3\sqrt[3]{x^2}}$; then $y^3 = 4+3x^{2/3}$ and $3y^2\,dy = 2x^{-1/3}\,dx$, or $\frac{3}{2}y^2\,dy = \dfrac{dx}{x^{1/3}}$. Also, $x^2 = \left(\frac{1}{3}(y^3-4)\right)^3 = \frac{1}{27}(y^3-4)^3$, and so

$$\int \frac{\sqrt[3]{(4+3\sqrt[3]{x^2})^5}}{\sqrt[3]{x^7}}\,dx = \int (4+3x^{2/3})^{5/3}\cdot\frac{1}{x^2}\cdot\frac{dx}{x^{1/3}} = \int y^5\cdot\frac{27}{(y^3-4)^3}\cdot\tfrac{3}{2}y^2\,dy = \int \frac{81y^7}{2(y^3-4)^3}\,dy.$$

Solution to exercise 4. — The integrand has infinite discontinuities at -3, -1, 0 and 2, the first three of which belong to the interval of integration. Thus,

$$\int_{-\infty}^{\frac{3}{2}} f = \lim_{t\to-\infty}\int_t^{-4} f + \lim_{t\to-3^-}\int_{-4}^{t} f + \lim_{t\to-3^+}\int_t^{-2} f + \lim_{t\to-1^-}\int_{-2}^{t} f + \lim_{t\to-1^+}\int_t^{-\frac{1}{2}} f + \lim_{t\to0^-}\int_{-\frac{1}{2}}^{t} f + \lim_{t\to0^+}\int_t^{\frac{3}{2}} f.$$

You may want to think about whether or not this integral is convergent.