Determine whether each series is convergent or divergent by applying the comparison test or the limit comparison test, with a geometric series, a p-series or a logarithmic p-series, and then completing the table below. The first column contains the series to be tested.

– Write the series used for comparison fully and explicitly in the second column, and write the relevant inequality satisfied by the parameter of the comparing series. The series used for comparison must be either a geometric series, or else a p-series, or else a logarithmic p-series. If possible, the series used for comparison must be optimal.

– If the comparison test is used, write a three-term inequality and a range of validity (which must include all sufficiently large positive integers) in the third column. The inequality must involve 0, the general term of the given series or its absolute value, a constant multiple of the general term of the series used for comparison, and nothing else.

– If the limit comparison test is used, write the value of the limit in the third column, and indicate the relevant inequality ($0 < \lambda$ or $\lambda < \infty$) satisfied by the limit.

– Write the conclusion, "C" for convergent, or "D" for divergent, in the fourth column.

Use the back of this page for rough work. (It may take any form you wish; it will not be checked.) Do not write anything but what is indicated above in the table. Marks are earned for entering correct items, as instructed above, in the table. For the cost of one mark per part, you may change the name of the variable in a given series (*e.g.*, replace k by n).

Series to be tested	Series used for comparison, and inequality satisfied by parameter	Inequality with range of validity, or limit with relevant inequality	Conclusion
a. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{3n^4+n-2}}{5n^3-3n+5}$			
b. $\displaystyle\sum_{k=2}^{\infty} \frac{(\log k)^2}{k^2}$			
c. $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(1/n)}{\sqrt{\log n}}$			
d. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{(\log k)^{7\log(\log(k))+1}}}$			
e. $\displaystyle\sum_{k=2}^{\infty} \frac{\pi - e\sin(k)}{k\log(k)\sqrt{\log k}}$			
f. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3\cdot 2^n-3}{7n^4+3^n}$			
g. $\displaystyle\sum_{k=1}^{\infty} \frac{e^{2\cos(k)}}{k^{2/3}}$			
h. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$			
i. $\displaystyle\sum_{j=1}^{\infty} \left(-\sqrt[5]{\log(\cos(1/j))}\right)$			
j. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{\sqrt{n}}\right)^n$			

Answers

Series to be tested	Series used for comparison, and inequality satisfied by parameter	Inequality with range of validity, or limit with relevant inequality	Conclusion
a. $\sum_{n=1}^{\infty} \frac{\sqrt[3]{3n^4+n-2}}{5n^3-3n+5}$	$\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}; \quad p>1$	$\lambda = \frac{1}{5}\sqrt[3]{3}; \quad \lambda < \infty.$	C
b. $\sum_{k=2}^{\infty} \frac{(\log k)^2}{k^2}$	$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}; \quad p>1$	$\lambda = 0; \quad \lambda < \infty.$	C
c. $\sum_{n=2}^{\infty} \frac{\sin(1/n)}{\sqrt{\log n}}$	$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\log n}}; \quad p \leqslant 1$	$0 < \frac{1}{2n\sqrt{\log n}} < \frac{\sin(1/n)}{\sqrt{\log n}}, \quad \text{if } n \geqslant 2.$	D
d. $\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{(\log k)^{7\log(\log(k))+1}}}$	$\sum_{k=2}^{\infty} \frac{1}{k(\log k)^{1/3}}; \quad p \leqslant 1$	$\lambda = \infty; \quad \lambda > 0$	D
e. $\sum_{k=2}^{\infty} \frac{\pi - e\sin(k)}{k\log(k)\sqrt{\log k}}$	$\sum_{k=2}^{\infty} \frac{1}{k(\log k)^{3/2}}; \quad p>1$	$0 < \frac{\pi - e\sin(k)}{k(\log k)^{3/2}} < \frac{\pi+e}{k(\log k)^{3/2}}, \quad \text{if } k \geqslant 2.$	C
f. $\sum_{n=1}^{\infty} \frac{2n^2+3\cdot 2^n-3}{7n^4+3^n}$	$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n; \quad 0<r<1.$	$\lambda = 3; \quad \lambda < \infty.$	C
g. $\sum_{k=1}^{\infty} \frac{e^{2\cos(k)}}{k^{2/3}}$	$\sum_{k=1}^{\infty} \frac{1}{k^{2/3}}; \quad p \leqslant 1$	$0 < \frac{e^{-2}}{k^{2/3}} \leqslant \frac{e^{2\cos(k)}}{k^{2/3}}, \quad \text{if } k \geqslant 1.$	D
h. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$	$\sum_{n=1}^{\infty} \frac{1}{n}; \quad p \leqslant 1$	$\frac{1}{n^{1+1/n}} > \frac{1}{en} > 0, \quad \text{if } n>1.$	D
i. $\sum_{j=1}^{\infty} \left(-\sqrt[5]{\log(\cos(1/j))}\right)$	$\sum_{j=1}^{\infty} \frac{1}{j^{2/5}}; \quad p \leqslant 1.$	$\lambda = 2^{-1/5}; \quad 0<\lambda.$	D
j. $\sum_{n=1}^{\infty} \left(1-\frac{1}{\sqrt{n}}\right)^n$	$\sum_{n=1}^{\infty} \frac{1}{n^2}; \quad p>1$	$\lambda = 0; \quad \lambda < \infty.$	C

a. If

$$a_n = \frac{\sqrt[3]{3n^4+n-2}}{5n^3-3n+5} \qquad \text{and} \qquad b_n = \frac{1}{n^{5/3}},$$

then $a_n > 0$ and $b_n > 0$ if $n \geqslant 1$, and

$$\lim \frac{a_n}{b_n} = \lim \frac{\sqrt[3]{3+1/n^3-2/n^4}}{5-3/n+5/n^2} = \tfrac{1}{5}\sqrt[3]{3}$$

is a positive real number. So the limit comparison test implies that the series $\sum a_n$ converges with the p-series $\sum b_n$ ($p = \frac{5}{3} > 1$).

b. If $k \geqslant 2$,

$$a_k = \frac{(\log k)^2}{k^2} \qquad \text{and} \qquad b_k = \frac{1}{k^{3/2}},$$

then $a_k > 0$ and $b_k > 0$. By elementary properties of the logarithm,

$$\lim \frac{a_k}{b_k} = \lim \frac{(\log k)^2}{k^{1/2}} = 0$$

is a non-negative real number. Therefore, the limit comparison test implies that the series $\sum a_k$ converges with the p-series $\sum b_k$ ($p = \frac{3}{2} > 1$).

c. Recall that $\vartheta\cos(\vartheta) < \sin(\vartheta) < \vartheta$ if $0 < \vartheta < \frac{1}{2}\pi$, and therefore $0 < 1/(2n) < \sin(1/n)$, provided $n \geqslant 1$. So if $n \geqslant 2$ then

$$0 < \frac{1}{2n\sqrt{\log n}} < \frac{\sin(1/n)}{\sqrt{\log n}}, \qquad \text{and thus} \qquad \sum_{n=2}^{\infty} \frac{\sin(1/n)}{\sqrt{\log n}}$$

diverges with the logarithmic p-series $\sum \left(n\sqrt{\log n}\right)^{-1}$ ($p = \frac{1}{2} \leqslant 1$), by the comparison test.

d. Let

$$a_k = \frac{1}{\sqrt[3]{(\log k)^{7\log(\log(k))+1}}} = \frac{e^{-\frac{7}{3}(\log(\log k))^2}}{(\log k)^{1/3}}$$

and

$$b_k = \frac{1}{k(\log k)^{1/3}};$$

then $a_k, b_k > 0$ if $k \geqslant 2$ and

$$\lim \frac{a_k}{b_k} = \lim \frac{k}{e^{\frac{7}{3}(\log(\log k))^2}} = \lim e^{\log(k) - \frac{7}{3}(\log(\log k))^2} = \infty,$$

since

$$\lim\left\{\log(k) - \tfrac{7}{3}(\log(\log k))^2\right\} = \lim_{x\to\infty}\left\{x\left(1 - \frac{7(\log x)^2}{3x}\right)\right\} = \infty,$$

where $x = \log k$, by elementary properties of the logarithm. Since $\lim(a_k/b_k)$ is infinite, the limit comparison test implies that the series $\sum a_k$ diverges with the logarithmic p-series $\sum b_k$ ($p = \frac{1}{3} \leqslant 1$).

e. If

$$a_k = \frac{\pi - e\sin(k)}{k\log(k)\sqrt{\log k}} \qquad \text{and} \qquad b_k = \frac{1}{k(\log k)^{3/2}},$$

then $0 < a_k < (\pi + e)b_k$, if $k \geqslant 2$, since $-1 < \sin(k) < 1$, and hence $0 < \pi - e < \pi - e\sin(k) < \pi + e$, for any integer k. Therefore, the comparison test implies that the series $\sum a_k$ converges with the logarithmic p-series $\sum b_k$ ($p = \frac{3}{2} > 1$).

f. If

$$a_n = \frac{2n^2 + 3\cdot 2^n - 3}{7n^4 + 3^n} \qquad \text{and} \qquad b_n = \left(\tfrac{2}{3}\right)^n,$$

then $a_n > 0$ and $b_n > 0$ if $n \geqslant 1$, and (by elementary properties of the exponential function),

$$\lim \frac{a_n}{b_n} = \lim \frac{(2n^2-3)/2^n + 3}{7n^4/3^n + 1} = 3$$

is a non-negative real number. Therefore, the limit comparison test implies that the series $\sum a_n$ converges with the geometric series $\sum b_n$ ($r = \frac{2}{3}$, so $0 < r < 1$).

g. If $k \geqslant 1$ then $0 < e^{-2}k^{-2/3} < e^{2\cos(k)}k^{-2/3}$, since $-1 < \cos(k)$, and hence $0 < e^{-2} < e^{2\cos(k)}$, for any positive integer k. So the comparison test implies that the series $\sum e^{2\cos(k)}k^{-2/3}$ diverges with the p-series $\sum k^{-2/3}$ ($p = \frac{2}{3} \leqslant 1$).

h. Since $1/x(x-1) < \log(x) < x - 1$ for $x > 1$, it follows that $0 < \log(n) < n$, *i.e.*, $0 < (\log n)/n < 1$, or $1 < n^{1/n} < e$, provided $n \geqslant 2$. Thus,

$$\frac{1}{n^{1+1/n}} = \frac{1}{n^{1/n}\cdot n} > \frac{1}{en} > 0.$$

Therefore, the comparison test implies that the series $\sum n^{-(1+1/n)}$ diverges with the harmonic series.

i. If

$$a_j = -\sqrt[5]{\log(\cos(1/j))} \qquad \text{and} \qquad b_j = \frac{1}{j^{2/5}},$$

then $a_j > 0$ and $b_j > 0$ if $j \geqslant 1$. Letting $x = 1/j$ and $y = \cos(x)$ gives

$$\lim\left(\frac{a_j}{b_j}\right)^5 = \lim_{y\to 1}\frac{\log(y)}{y-1}\cdot\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \tfrac{1}{2},$$

in which

$$\lim_{y\to 1}\frac{\log(y)}{y-1} = 1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \tfrac{1}{2}$$

(the first by elementary properties of the logarithm, the second using $1 - \cos(x) = 2\sin^2\left(\frac{1}{2}x\right)$ and $(\sin\vartheta)/\vartheta \to 1$ as $\vartheta \to 0$). Since $\lim(a_j/b_j) = 2^{-1/5} > 0$, the limit comparison test implies that the series $\sum a_j$ diverges with the p-series $\sum b_j$ ($p = \frac{2}{5} \leqslant 1$).

j. If

$$a_n = \left(1 - \frac{1}{\sqrt{n}}\right)^n \qquad \text{and} \qquad b_n = \frac{1}{n^2},$$

then $a_n > 0$ and $b_n > 0$ if $n > 1$, and

$$\lim \frac{a_n}{b_n} = \lim\left\{\left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} n^{2/\sqrt{n}}\right\}^{\sqrt{n}} = 0,$$

since $n^{2/\sqrt{n}} = e^{2(\log n)/\sqrt{n}}$ and

$$\lim\left\{\left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}} e^{2(\log n)/\sqrt{n}}\right\} = e^0\cdot e^{-1} = \frac{1}{e} < 1,$$

by elementary properties of the logarithm. Since $\lim(a_n/b_n)$ is a non-negative real number, the limit comparison test implies that $\sum a_n$ converges with the p-series $\sum b_n$ ($p = 2 > 1$).