Review exercises

Exercise 1. Evaluate each of the following integrals.

a. $\displaystyle\int_1^3 \frac{dx}{\sqrt{4x-x^2}}$ b. $\displaystyle\int_{-2}^2 (2x-3)\sqrt{4-x^2}\,dx$ c. $\displaystyle\int \frac{dw}{\sqrt{4w^2+3}}$

d. $\displaystyle\int \frac{\sec(1/x)}{x^2}\,dx$ e. $\displaystyle\int_0^{2\pi} \sqrt{1-\cos y}\,dy$ f. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

g. $\displaystyle\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}}$ h. $\displaystyle\int_{-2}^{-\frac{2}{3}\sqrt{3}} z\operatorname{arcsec} z\,dz$ i. $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \frac{\log(\tan x)}{\sin x\cos x}\,dx$

j. $\displaystyle\int t^2\cos(2t)\,dt$ k. $\displaystyle\int_{-2}^0 \arctan(\sqrt{1-x})\,dx$ l. $\displaystyle\int \sin(\log r)\,dr$

m. $\displaystyle\int \sin(\sqrt[3]{x})\,dx$ n. $\displaystyle\int \sin(2x)\sin(3x)\,dx$ o. $\displaystyle\int \frac{\sin(\log r)}{r}\,dr$

Exercise 2. Evaluate each of the following integrals.

a. $\displaystyle\int e^{\alpha x}\sin(\beta x)\,dx$ b. $\displaystyle\int_0^1 \frac{2x-3}{\sqrt{4-x^2}}\,dx$ c. $\displaystyle\int \frac{dw}{w^9-\pi w}$

d. $\displaystyle\int z(\operatorname{arcsec} z)^2\,dz$ e. $\displaystyle\int \frac{dy}{(9y^2-42y+49)^2}$ f. $\displaystyle\int \frac{3x+7}{\sqrt{2x^2+5x-1}}\,dx$

g. $\displaystyle\int_0^1 \frac{te^{-\frac{1}{2}t}}{(2-t)^2}\,dt$ h. $\displaystyle\int \frac{3x+2}{\sqrt{5+4x-x^2}}\,dx$ i. $\displaystyle\int_{\frac{5}{3}}^5 \arcsin\left(\sqrt{\frac{x}{x+5}}\right)dx$

j. $\displaystyle\int \frac{\sqrt{1+e^x}}{e^x}\,dx$ k. $\displaystyle\int_0^{\frac{1}{4}} \arcsin(\sqrt{x})\,dx$ l. $\displaystyle\int \frac{2x-1}{5x^2+x+2}\,dx$

m. $\displaystyle\int \frac{1+\sin\vartheta}{\vartheta-\cos\vartheta}\,d\vartheta$ n. $\displaystyle\int \frac{\vartheta-\cos\vartheta}{1+\sin\vartheta}\,d\vartheta$ o. $\displaystyle\int e^{\sin t}\cdot\frac{t\cos^3 t-\sin t}{\cos^2 t}\,dt$

Exercise 3. a. Verify the identity

$$2\arctan\left(\frac{x}{1+\sqrt{1-x^2}}\right)=\arcsin x, \qquad \text{for} \quad -1\leqslant x\leqslant 1.$$

b. Show that there is a real number α such that

$$\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=\alpha+\arcsin\sqrt{\frac{x}{x+1}} \qquad \text{if } x>1,$$

and then determine the value of α.

Exercise 4. Suppose that f is continuously differentiable on $\mathbb{R}$, and that

$$f(1)=3, \qquad f(4)=-2, \qquad \text{and} \qquad \int_1^4 f(x)\,dx=4.$$

a. Evaluate $\displaystyle\int_{\frac{7}{6}\pi}^{\frac{1}{2}\pi} f(4\sin^2(\vartheta))\sin(2\vartheta)\,d\vartheta$. b. Evaluate $\displaystyle\int_{-2}^1 t^3 f'(t^2)\,dt$.

Exercise 5. Evaluate each definite integral.

a. $\displaystyle\int_0^{\frac{1}{3}} \arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)dx$ b. $\displaystyle\int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{\log(2x)}{x^2+1}\,dx$

Exercise 6. Use partial integration to derive a reduction formula for

$$\int e^{\alpha x}\sin^n(x)\,dx,$$

and use this formula to compute the integrals

$$\int_0^{\frac{1}{2}\pi} e^x\sin^3(x)\,dx \qquad \text{and} \qquad \int_0^{\pi} e^{-x}\sin^4(x)\,dx.$$

A sample Test

Question 1. a. Show that

$$\frac{d}{dz}\left\{-\frac{\arcsin(z)}{z}-\log\left|\frac{1+\sqrt{1-z^2}}{z}\right|\right\}=\frac{\arcsin(z)}{z^2},$$

and use it to compute the definite integral

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\arcsin(z)}{z^2}\,dz.$$

b. Evaluate and simplify each of the following.
i. $\sin\big(\arctan(2/x)\big)$ ii. $\arctan\big(\cot(-1/x)\big)$, for $x>1$ iii. $\displaystyle\lim_{y\to 0^+}\operatorname{arcsec}(\log y)$.

Question 2. Evaluate each of the following integrals. Use the variable names as given. You should write the answers only, and simplify all numerical coefficients. Try to not spend more than a few minutes on this question.

a. $\displaystyle\int \sqrt[4]{6x+7}\,dx$ b. $\displaystyle\int \frac{14\,dw}{\sqrt{(7w-3)^2+21}}$ c. $\displaystyle\int \cos(3y-5)\,dy$ d. $\displaystyle\int \sec(2z+1)\,dz$

Question 3. Evaluate each integral and simplify the result. Use the variable names as given. To earn full credit for part a you must change the limits of integration whenever you change the variable of integration.

a. $\displaystyle\int_5^{13} \frac{dw}{(w+3)\sqrt{w-1}}$ b. $\displaystyle\int \frac{(v-1)\sin(ve^{-v})}{e^v\cos(ve^{-v})}\,dv$ c. $\displaystyle\int \frac{dx}{e^x-4e^{-x}}$

Question 4. Evaluate each integral and simplify the result. Use the variable names as given.

a. $\displaystyle\int \sin(3\vartheta)\cos(4\vartheta)\,d\vartheta$ b. $\displaystyle\int_1^e \left(\frac{\ln y}{y}\right)^2 dy$ c. $\displaystyle\int (2t-1)^2 e^{-4t}\,dt$

Question 5. Evaluate each integral and simplify the result. Use the variable names as given.

a. $\displaystyle\int e^{6x}\cos(e^{2x}+3)\,dx$ b. $\displaystyle\int \frac{3x+4}{\sqrt{13-8x-5x^2}}\,dx$ c. $\displaystyle\int \frac{\sqrt{1+\ln x}}{x(\ln x)^2}\,dx$

d. $\displaystyle\int \frac{\sqrt{9x^2-5}}{x^2}\,dx$ e. $\displaystyle\int_{\frac13\sqrt3}^{\sqrt3} \arctan(1/x)\,dx$ f. $\displaystyle\int_0^{\frac12} \ln\left(\frac{1+x}{1-x}\right)dx$

Question 6. Attempt **one** part of this question.

a. Use symmetry to evaluate the definite integral

$$\int_1^3 \sin\left(\tfrac14\pi x\right)\ln\left(x-2+\sqrt{x^2-4x+8}\right)dx.$$

b. Use partial integration to derive a reduction formula which expresses

$$\int \frac{dt}{(\alpha t^2+\beta)^{n+1}}, \qquad \text{in terms of} \qquad \int \frac{dt}{(\alpha t^2+\beta)^n}.$$

Use this reduction formula to compute

$$\int \frac{dx}{(x^2+2x+5)^{5/2}}.$$

Solutions to the review exercises

1. a. Completing the square gives

$$\int_1^3 \frac{dx}{\sqrt{4x-x^2}} = \int_1^3 \frac{dx}{\sqrt{4-(x-2)^2}} = \arcsin\left(\tfrac12(x-2)\right)\Big|_1^3 = \tfrac13\pi.$$

b. Separating the integral into a difference gives

$$\int_{-2}^2 (2x-3)\sqrt{4-x^2}\,dx = 2\int_{-2}^2 x\sqrt{4-x^2}\,dx - 3\int_{-2}^2 \sqrt{4-x^2}\,dx = -6\pi,$$

since the first term is the integral of an odd function, and the second is the area of semicircle of radius 2.

c. Using a standard integral gives

$$\int \frac{dw}{\sqrt{4w^2+3}} = \tfrac12\log\left|2w+\sqrt{4w^2+3}\right| + C.$$

d. If $t=1/x$, then $dt = -\,dx/x^2$, so

$$\int \frac{\sec(1/x)}{x^2}\,dx = -\int \sec(t)\,dt = -\log|\sec(1/x)+\tan(1/x)| + C.$$

e. The halving identity for the sine function implies that

$$\sqrt{1-\cos y} = \sqrt{2\sin^2\left(\tfrac12 y\right)} = \sqrt2\,\sin\left(\tfrac12 y\right),$$

since $\sin\left(\tfrac12 y\right) \geqslant 0$ on $[0,2\pi]$. Therefore,

$$\int_0^{2\pi} \sqrt{1-\cos y}\,dy = \sqrt2\int_0^{2\pi} \sin\left(\tfrac12 y\right)dy = -2\sqrt2\cos\left(\tfrac12 y\right)\Big|_0^{2\pi} = 4\sqrt2.$$

f. Let $t=e^x$, or $\log t = x$, so that $dt/t = dx$; then

$$\int \frac{dx}{e^x+e^{-x}} = \int \frac{dt}{t^2+1} = \arctan(t)+C = \arctan(e^x)+C.$$

g. Rationalizing the denominator gives

$$\int \frac{dx}{\sqrt{x+1}+\sqrt{x-1}} = \tfrac12\int\left(\sqrt{x+1}-\sqrt{x-1}\right)dx = \tfrac13(x+1)^{3/2} - \tfrac13(x-1)^{3/2} + C.$$

h. Integrating by parts, and then (implicitly) letting $t=\sqrt{z^2-1}$, gives

$$\begin{aligned}\int_{-2}^{-\frac23\sqrt3} z\,\mathrm{arcsec}(z)\,dz &= \tfrac12 z^2\,\mathrm{arcsec}(z)\Big|_{-2}^{-\frac23\sqrt3} - \tfrac12\int_{-2}^{-\frac23\sqrt3} \frac{z}{\sqrt{z^2-1}}\,dz\\ &= \left(\tfrac79\pi - \tfrac83\pi\right) - \tfrac12\sqrt{z^2-1}\,\Big|_{-2}^{-\frac23\sqrt3}\\ &= -\tfrac{17}{9}\pi + \tfrac13\sqrt3.\end{aligned}$$

i. If $t=\log(\tan x)$, then $dt = \dfrac{\sec^2(x)}{\tan x}\,dx = \dfrac{dx}{\sin x\cos x}$, and thus

$$\int_{\frac14\pi}^{\frac13\pi} \frac{\log(\tan x)}{\sin x\cos x}\,dx = \int_0^{\frac12\log 3} t\,dt = \tfrac12 t^2\Big|_0^{\frac12\log 3} = \tfrac18(\log 3)^2.$$

j. Repeated partial integration yields

$$\begin{aligned}\int t^2\cos 2t\,dt &= \left(t^2\right)\left(\tfrac12\sin 2t\right) - \left(2t\right)\left(-\tfrac14\cos 2t\right) + \left(2\right)\left(-\tfrac18\sin 2t\right) + C\\ &= \tfrac14(2t^2-1)\sin 2t + \tfrac12 t\cos 2t + C.\end{aligned}$$

k. The derivative of the integrand is
$$\frac{d}{dx}\left\{\arctan\sqrt{1-x}\right\} = \frac{1}{1+(1-x)}\cdot\frac{-1}{2\sqrt{1-x}} = \frac{1}{2(x-2)\sqrt{1-x}},$$
so partial integration, with the primitive function $x-2$ of 1, gives
$$\begin{aligned}\int_{-2}^{0}\arctan(\sqrt{1-x})\,dx &= (x-2)\arctan(\sqrt{1-x})\Big|_{-2}^{0} - \tfrac{1}{2}\int_{-2}^{0}\frac{dx}{\sqrt{1-x}}\\ &= \left(-\tfrac{1}{2}\pi-(-\tfrac{4}{3}\pi)\right)+\sqrt{1-x}\Big|_{-2}^{0}\\ &= \tfrac{5}{6}\pi-\sqrt{3}+1.\end{aligned}$$

l. Repeated partial integration yields
$$\int\sin(\log r)\,dr = r\sin(\log r)-\int\cos(\log r)\,dr = r\sin(\log r)-r\cos(\log r)-\int\sin(\log r)\,dr,$$
and therefore (solving this equation for the integral in question),
$$\int\sin(\log r)\,dr = \tfrac{1}{2}r(\sin(\log r)-\cos(\log r))+C.$$

m. If $t=\sqrt[3]{x}$ then $t^3=x$, and $3t^2\,dt=dx$, and then repeated partial integration gives
$$\begin{aligned}\int\sin(\sqrt[3]{x})\,dx &= 3\int t^2\sin t\,dt = (3t^2)(-\cos t)-(6t)(-\sin t)+(6)(\cos t)+C\\ &= 3(2-t^2)\cos t+6t\sin t+C = 3(2-\sqrt[3]{x^2})\cos(\sqrt[3]{x})+6\sqrt[3]{x}\sin(\sqrt[3]{x})+C.\end{aligned}$$

n. Repeated partial integration yields
$$\begin{aligned}\int\sin(2x)\sin(3x)\,dx &= -\tfrac{1}{2}\cos(2x)\sin(3x)+\tfrac{3}{2}\int\cos(2x)\cos(3x)\,dx\\ &= -\tfrac{1}{2}\cos(2x)\sin(3x)+\tfrac{3}{4}\sin(2x)\cos(3x)+\tfrac{9}{4}\int\sin(2x)\sin(3x)\,dx.\end{aligned}$$
Solving this last equation for the integral in question yields
$$\int\sin(2x)\sin(3x)\,dx = \tfrac{2}{5}\cos(2x)\sin(3x)-\tfrac{3}{5}\sin(2x)\cos(3x)+C.$$

o. Let $t=\log r$, so that $dt=dr/r$, and hence
$$\int\frac{\sin(\log r)}{r}\,dr = \int\sin(t)\,dt = -\cos(t)+C = -\cos(\log r)+C.$$

2. a. Repeated partial integration, integrating the exponential factor and differentiating the trigonometric factor, gives
$$\int e^{\alpha x}\sin(\beta x)\,dx = \frac{1}{\alpha}e^{\alpha x}\sin(\beta x)-\frac{\beta}{\alpha^2}e^{\alpha x}\cos(\beta x)-\frac{\beta^2}{\alpha^2}\int e^{\alpha x}\sin(\beta x)\,dx,$$
or
$$\frac{\alpha^2+\beta^2}{\alpha^2}\int e^{\alpha x}\sin(\beta x)\,dx = \frac{1}{\alpha^2}e^{\alpha x}(\alpha\sin(\beta x)-\beta\cos(\beta x))+C,$$
and so
$$\int e^{\alpha x}\sin(\beta x)\,dx = \frac{e^{\alpha x}(\alpha\sin(\beta x)-\beta\cos(\beta x))}{\alpha^2+\beta^2}+C.$$

b. Integrating term-by-term gives
$$\int_0^1\frac{2x-3}{\sqrt{4-x^2}}\,dx = \int_0^1\frac{2x\,dx}{\sqrt{4-x^2}}-\int_0^1\frac{3\,dx}{\sqrt{4-x^2}} = -2\sqrt{4-x^2}\Big|_0^1-3\arcsin\left(\tfrac{1}{2}x\right)\Big|_0^1 = 4-2\sqrt{3}-\tfrac{1}{2}\pi.$$

c. If $x=1-\pi w^{-8}$, then $dx=8\pi w^{-9}\,dw$, and so
$$\begin{aligned}\int\frac{dw}{w^9-\pi w} &= \int\frac{1}{1-\pi w^{-8}}\cdot\frac{dw}{w^9} = \frac{1}{8\pi}\int\frac{dx}{x} = \frac{1}{8\pi}\log|x|+k\\ &= \frac{1}{8\pi}\log\left|1-\pi w^{-8}\right|+k.\end{aligned}$$

d. Partial integration, integrating the power and differentiating the inverse secant, gives
$$\int z(\operatorname{arcsec} z)^2\,dz = \tfrac{1}{2}z^2(\operatorname{arcsec} z)^2-\int\frac{z^2\operatorname{arcsec} z}{z\sqrt{z^2-1}}\,dz.$$
Cancelling the common factor z in the integrand on the right and integrating by parts, integrating the algebraic factor and differentiating the inverse secant, gives
$$\int(\operatorname{arcsec} z)\cdot\frac{z\,dz}{\sqrt{z^2-1}} = (\operatorname{arcsec} z)\sqrt{z^2-1}-\int\frac{\sqrt{z^2-1}}{z\sqrt{z^2-1}}\,dz,$$
where the integral on the right is evidently $\log|z|$. Therefore,
$$\int z(\operatorname{arcsec} z)^2\,dz = \tfrac{1}{2}(z\operatorname{arcsec} z)^2-(\operatorname{arcsec} z)\sqrt{z^2-1}+\log|z|+C.$$

e. Since $9y^2-42y+49$ is equal to $(3y-7)^2$, the integral in question is equal to
$$\int\frac{dy}{(3y-7)^4} = -\frac{1}{9(3y-7)^3}+C.$$

f. Notice that $2x^2+5x-1=\frac{1}{8}((4x+5)-33)$, and $3x+7=\frac{3}{4}(4x+5)+\frac{13}{4}$, so that
$$\int\frac{3x+7}{\sqrt{2x^2+5x-1}}\,dx = \frac{3}{4}\int\frac{4x+5}{\sqrt{2x^2+5x-1}}\,dx+\frac{13}{4}\int\frac{dx}{\sqrt{2x^2+5x-1}},$$
and by inspection,
$$\int\frac{4x+5}{\sqrt{2x^2+5x-1}}\,dx = 2\sqrt{2x^2+5x-1}+C.$$
Next, multiplying and dividing by $\sqrt{8}=2\sqrt{2}$ gives
$$\begin{aligned}\int\frac{dx}{\sqrt{2x^2+5x-1}} &= 2\sqrt{2}\int\frac{dx}{\sqrt{(4x+5)^2-33}}\\ &= \tfrac{1}{2}\sqrt{2}\ln\left|4x+5+2\sqrt{2}\sqrt{2x^2+5x-1}\right|+C\end{aligned}$$
Therefore, the integral in question is equal to
$$\tfrac{3}{2}\sqrt{2x^2+5x-1}+\tfrac{13}{8}\sqrt{2}\ln\left|4x+5+2\sqrt{2}\sqrt{2x^2+5x-1}\right|+C.$$

g. Since

$$\frac{d}{dt}\left\{te^{-\frac{1}{2}t}\right\} = e^{-\frac{1}{2}t} + te^{-\frac{1}{2}t}\left(-\tfrac{1}{2}\right) = \tfrac{1}{2}e^{-\frac{1}{2}t}(2-t),$$

partial integration, integrating the negative power of $2-t$ and differentiating the numerator, gives

$$\int_0^1 \frac{te^{-\frac{1}{2}t}}{(2-t)^2}\,dt = \frac{te^{-\frac{1}{2}t}}{2-t}\bigg|_0^1 - \frac{1}{2}\int_0^1 e^{-\frac{1}{2}t}\,dt = e^{-1/2} + e^{-\frac{1}{2}t}\bigg|_0^1$$
$$= 2e^{-1/2} - 1.$$

h. Since $5+4x-x^2 = 9-(x-2)^2$, and $3x+2 = 3(x-2)+8$, it follows that

$$\int \frac{3x+2}{\sqrt{5+4x-x^2}}\,dx = 3\int \frac{x-2}{\sqrt{5+4x-x^2}}\,dx + 8\int \frac{dx}{\sqrt{9-(x-2)^2}}$$
$$= -3\sqrt{5+4x-x^2} + 8\arcsin\left(\tfrac{1}{3}(x-2)\right) + C.$$

i. If $x > 0$ then

$$\frac{d}{dx}\left\{\arcsin\left(\sqrt{\frac{x}{x+5}}\right)\right\} = \frac{1}{\sqrt{1-\frac{x}{x+5}}}\cdot\frac{\sqrt{x+5}}{2\sqrt{x}}\cdot\frac{5}{(x+5)^2} = \frac{\sqrt{5}}{2(x+5)\sqrt{x}},$$

so partial integration gives

$$\int_{\frac{5}{3}}^{5} \arcsin\left(\sqrt{\frac{x}{x+5}}\right)dx = (x+5)\arcsin\left(\sqrt{\frac{x}{x+5}}\right)\bigg|_{\frac{5}{3}}^{5} - \frac{1}{2}\sqrt{5}\int_{\frac{5}{3}}^{5}\frac{dx}{\sqrt{x}}$$
$$= \frac{5}{2}\pi - \frac{10}{9}\pi - \sqrt{5x}\,\bigg|_{\frac{5}{3}}^{5}$$
$$= \frac{25}{18}\pi - 5 + \frac{5}{3}\sqrt{3}.$$

j. If $t = \sqrt{1+e^x}$, then

$$e^x = t^2 - 1, \qquad \text{or} \qquad x = \log(t^2-1), \qquad \text{and} \qquad dx = \frac{2t}{t^2-1}\,dt.$$

Expressing the integral in terms of t and the integrating by parts (integrating the second factor and differentiating t), gives

$$\int \frac{\sqrt{1+e^x}}{e^x}\,dx = \int t\cdot\frac{2t\,dt}{(t^2-1)^2}\,dt = -\frac{t}{t^2-1} + \int\frac{dt}{t^2-1} = -\frac{t}{t^2-1} + \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$
$$= -\frac{\sqrt{1+e^x}}{e^x} + \frac{1}{2}\log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C = -\frac{\sqrt{1+e^x}}{e^x} - \log\left(1+\sqrt{1+e^x}\right) + C.$$

k. If $t = \arcsin(\sqrt{x})$, then $x = \sin^2 t$ and $dx = 2\sin t\cos t\,dt = \sin(2t)\,dt$, so partial integration gives

$$\int_0^{\frac{1}{4}} \arcsin(\sqrt{x})\,dx = \int_0^{\frac{1}{6}\pi} t\sin(2t)\,dt = \left(-\tfrac{1}{2}t\cos(2t) + \tfrac{1}{4}\sin(2t)\right)\bigg|_0^{\frac{1}{6}\pi} = -\frac{1}{24}\pi + \frac{1}{8}\sqrt{3}.$$

Note: One could instead integrate by parts directly, after which a different change of variables would be appropriate.

l. Since

$$5x^2+x+2 = \tfrac{1}{20}\left((10x+1)^2+39\right) \qquad \text{and} \qquad 2x-1 = \tfrac{1}{5}\left((10x+1)-6\right),$$

it follows that

$$\int\frac{2x-1}{5x^2+x+2}\,dx = \frac{1}{5}\int\frac{10x+1}{5x^2+x+2}\,dx - 24\int\frac{dx}{(10x+1)^2+39}$$
$$= \tfrac{1}{5}\ln|5x^2+x+2| - \tfrac{4}{65}\sqrt{39}\arctan\left(\tfrac{1}{39}(10x+1)\sqrt{39}\right) + C.$$

(Note that $\frac{24}{10\sqrt{39}} = \frac{4}{65}\sqrt{39}$.)

m. If $t = \vartheta - \cos\vartheta$, then $dt = (1+\sin\vartheta)\,d\vartheta$, and hence

$$\int\frac{1+\sin\vartheta}{\vartheta-\cos\vartheta}\,d\vartheta = \int\frac{dt}{t} = \log|t| + C = \log|\vartheta-\cos\vartheta| + C.$$

n. Multiplying and dividing the integrand by $1-\sin\vartheta$ gives

$$\int\frac{\vartheta-\cos\vartheta}{1+\sin\vartheta}\cdot\frac{1-\sin\vartheta}{1-\sin\vartheta}\,d\vartheta = \int\frac{\vartheta-\cos\vartheta-\vartheta\sin\vartheta+\cos\vartheta\sin\vartheta}{\cos^2\vartheta}\,d\vartheta$$
$$= \int\vartheta\sec^2\vartheta\,d\vartheta - \int\sec\vartheta\,d\vartheta - \int\vartheta\sec\vartheta\tan\vartheta\,d\vartheta + \int\tan\vartheta\,d\vartheta.$$

Partial integration then gives

$$\int\vartheta\sec^2\vartheta\,d\vartheta = \vartheta\tan\vartheta - \int\tan\vartheta\,d\vartheta \qquad \text{and} \qquad \int\vartheta\sec\vartheta\tan\vartheta\,d\vartheta = \vartheta\sec\vartheta - \int\sec\vartheta\,d\vartheta.$$

Therefore,

$$\int\frac{\vartheta-\cos\vartheta}{1+\sin\vartheta}\,d\vartheta = \vartheta(\tan\vartheta - \sec\vartheta) + C.$$

o. Expanding the integrand, and then integrating each term by parts, gives

$$\int e^{\sin t}\cdot\frac{t\cos^3 t - \sin t}{\cos^2 t}\,dt = \int te^{\sin t}\cos t\,dt - \int e^{\sin t}\sec t\tan t\,dt$$
$$= te^{\sin t} - \int e^{\sin t}\,dt - e^{\sin t}\sec t + \int e^{\sin t}\,dt$$
$$= e^{\sin t}(t - \sec t) + C.$$

3. a. Let

$$\vartheta = \arctan\left(\frac{x}{1+\sqrt{1-x^2}}\right), \quad \text{so} \quad \tan\vartheta = \frac{x}{1+\sqrt{1-x^2}}.$$

If $-1 \leqslant x \leqslant 1$ then $-1 \leqslant x/(1+\sqrt{1-x^2}) \leqslant 1$, and so 2ϑ lies in $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$, on which the sine function is one-to-one. Since

$$\begin{aligned}\sin 2\vartheta = 2\sin\vartheta\cos\vartheta &= \frac{2\tan\vartheta}{1+\tan^2\vartheta}\\ &= \frac{2x}{\left(1+\sqrt{1-x^2}\right)\left(1+\left(x/(1+\sqrt{1-x^2})\right)^2\right)}\\ &= \frac{2x(1+\sqrt{1-x^2})}{(1+\sqrt{1-x^2})^2+x^2} = \frac{2x(1+\sqrt{1-x^2})}{1+2\sqrt{1-x^2}+1-x^2+x^2}\\ &= x,\end{aligned}$$

the identity follows. (Alternatively, the identity could be proved as in part b, by differentiating and applying the mean value theorem.)

b. Using the chain and quotient Rules, and then simplifying, gives

$$\begin{aligned}\frac{d}{dx}\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) &= \frac{1}{1+\dfrac{(1+\sqrt{x})^2}{(1-\sqrt{x})^2}}\cdot\frac{\dfrac{1}{2\sqrt{x}}(1-\sqrt{x})-(1+\sqrt{x})\dfrac{-1}{2\sqrt{x}}}{(1-\sqrt{x})^2}\\ &= \frac{1}{\left((1-\sqrt{x})^2+(1+\sqrt{x})^2\right)\sqrt{x}} = \frac{1}{2(1+x)\sqrt{x}},\end{aligned}$$

provided $x>0$ and $x\neq 1$, and

$$\begin{aligned}\frac{d}{dx}\arcsin\left(\sqrt{\frac{x}{x+1}}\right) &= \frac{1}{\sqrt{1-\dfrac{x}{x+1}}}\cdot\frac{\sqrt{x+1}}{2\sqrt{x}}\cdot\frac{1}{(x+1)^2}\\ &= \sqrt{1+x}\cdot\frac{\sqrt{x+1}}{2\sqrt{x}}\cdot\frac{1}{(x+1)^2} = \frac{1}{2(1+x)\sqrt{x}},\end{aligned}$$

provided $x>0$. So the mean value theorem implies that there is a real number α such that

$$\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \alpha + \arcsin\left(\sqrt{\frac{x}{x+1}}\right)$$

if $x>1$. Since

$$\lim_{x\to\infty}\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \arctan(-1) = -\tfrac{1}{4}\pi$$

and

$$\lim_{x\to\infty}\arcsin\left(\sqrt{\frac{x}{x+1}}\right) = \arcsin(1) = \tfrac{1}{2}\pi,$$

it follows that $\alpha = -\frac{3}{4}\pi$.

4. a. Let $x = 4\sin^2\vartheta$; then $dx = 8\sin\vartheta\cos\vartheta\,d\vartheta = 4\sin 2\vartheta\,d\vartheta$, and so

$$\int_{\frac{7}{6}\pi}^{\frac{1}{2}\pi} f(4\sin^2\vartheta)\sin 2\vartheta\,d\vartheta = \tfrac{1}{4}\int_1^4 f(x)\,dx = 1.$$

b. If $x=t^2$ then $dx = 2t\,dt$; partial integration (integrating f' and differentiating the power) then gives

$$\begin{aligned}\int_{-2}^{1} t^3 f'(t^2)\,dt = \tfrac{1}{2}\int_4^1 x f'(x)\,dx &= \tfrac{1}{2}xf(x)\Big|_4^1 - \tfrac{1}{2}\int_4^1 f(x)\,dx\\ &= \tfrac{1}{2}(3+8)-\tfrac{1}{2}(-4)\\ &= \tfrac{15}{2}.\end{aligned}$$

5. a. Since

$$\begin{aligned}\frac{d}{dx}\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) &= \frac{1}{1+\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)^2}\cdot\frac{(1-\sqrt{x})/(2\sqrt{x})+(1+\sqrt{x})/(2\sqrt{x})}{(1-\sqrt{x})^2}\\ &= \frac{1}{\left((1-\sqrt{x})^2+(1+\sqrt{x})^2\right)\sqrt{x}} = \frac{1}{2(x+1)\sqrt{x}},\end{aligned}$$

partial integration, with the primitive $x+1$ of 1, yields

$$\int_0^{\frac{1}{3}}\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)dx = (x+1)\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\Bigg|_0^{\frac{1}{3}} - \frac{1}{2}\int_0^{\frac{1}{3}}\frac{dx}{\sqrt{x}}.$$

Now

$$\frac{1}{2}\int_0^{\frac{1}{3}}\frac{dx}{\sqrt{x}} = \sqrt{x}\Big|_0^{\frac{1}{3}} = \tfrac{1}{3}\sqrt{3},$$

so it remains to evaluate

$$(x+1)\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\Bigg|_0^{\frac{1}{3}} = \tfrac{4}{3}\arctan\left(\frac{1+\frac{1}{3}\sqrt{3}}{1-\frac{1}{3}\sqrt{3}}\right) - \tfrac{1}{4}\pi.$$

Recall that

$$\arctan\left(\frac{1+\frac{1}{3}\sqrt{3}}{1-\frac{1}{3}\sqrt{3}}\right) = \arctan 1 + \arctan\tfrac{1}{3}\sqrt{3} = \tfrac{1}{4}\pi + \tfrac{1}{6}\pi = \tfrac{5}{12}\pi.$$

Therefore,

$$\int_0^{\frac{1}{3}}\arctan\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)dx = \tfrac{4}{3}\cdot\tfrac{5}{12}\pi - \tfrac{1}{4}\pi - \tfrac{1}{3}\sqrt{3} = \tfrac{11}{36}\pi - \tfrac{1}{3}\sqrt{3}.$$

b. Separating the integrand gives

$$\int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}}\frac{\log 2x}{x^2+1}\,dx = \int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}}\frac{\log 2}{x^2+1}\,dx + \int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}}\frac{\log x}{x^2+1}\,dx.$$

If $x = 1/t$, then $dx = -dt/t^2$ and $\log x = -\log t$; so

$$\int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{\log x}{x^2+1}\,dx = -\int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{\log t}{t^2+1}\,dt, \quad \text{and hence} \quad \int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{\log x}{x^2+1}\,dx = 0$$

Therefore,

$$\int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{\log 2x}{x^2+1}\,dx = \int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{\log 2}{x^2+1}\,dx = (\log 2)(\arctan t)\Big|_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} = \tfrac{1}{6}\pi \log 2.$$

6. Partial integration, integrating the exponential factor and differentiating the trigonometric factor, gives

$$\mathcal{I}_n = \int e^{\alpha x}\sin^n(x)\,dx = \frac{1}{\alpha}e^{\alpha x}\sin^n(x) - \frac{n}{\alpha}\int e^{\alpha x}\sin^{n-1}(x)\cos(x)\,dx. \qquad (\dagger)$$

Since

$$\begin{aligned}\frac{d}{dx}\Big\{\sin^{n-1}(x)\cos(x)\Big\} &= (n-1)\sin^{n-2}x\cos^2(x) - \sin^n(x)\\ &= (n-1)\sin^{n-2}(x)(1-\sin^2(x)) - \sin^n(x)\\ &= (n-1)\sin^{n-2}(x) - n\sin^n(x),\end{aligned}$$

applying the same strategy to the remaining integral gives

$$\begin{aligned}\int e^{\alpha x}\sin^{n-1}x\cos x\,dx &= \frac{1}{\alpha}e^{\alpha x}\sin^{n-1}x\cos x - \frac{1}{\alpha}\int e^{\alpha x}\Big((n-1)\sin^{n-2}x - n\sin^n x\Big)dx\\ &= \frac{1}{\alpha}e^{\alpha x}\sin^{n-1}x\cos x - \frac{n-1}{\alpha}\mathcal{I}_{n-2} + \frac{n}{\alpha}\mathcal{I}_n.\end{aligned}$$

Replacing the remaining integral in (†) by this last expression, and simplifying, gives

$$\mathcal{I}_n = \frac{1}{\alpha^2}e^{\alpha x}\sin^{n-1}x(\alpha\sin x - n\cos x) + \frac{n(n-1)}{\alpha^2}\mathcal{I}_{n-2} - \frac{n^2}{\alpha^2}\mathcal{I}_n,$$

or

$$\frac{\alpha^2+n^2}{\alpha^2}\mathcal{I}_n = \frac{1}{\alpha^2}e^{\alpha x}\sin^{n-1}x(\alpha\sin x - n\cos x) + \frac{n(n-1)}{\alpha^2}\mathcal{I}_{n-2},$$

and therefore

$$\mathcal{I}_n = \frac{e^{\alpha x}\sin^{n-1}x}{\alpha^2+n^2}(\alpha\sin x - n\cos x) + \frac{n(n-1)}{\alpha^2+n^2}\mathcal{I}_{n-2}.$$

Applying the reduction formula to the first integral gives

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} e^x\sin^3 x\,dx &= \tfrac{1}{10}e^x\sin^2 x(\sin x - 3\cos x)\Big|_0^{\frac{1}{2}\pi} + \tfrac{3}{5}\int_0^{\frac{1}{2}\pi} e^x\sin x\,dx\\ &= \tfrac{1}{10}e^{\frac{1}{2}\pi} + \tfrac{3}{5}\int_0^{\frac{1}{2}\pi} e^x\sin x\,dx.\end{aligned}$$

Next, integrating by parts gives

$$\int_0^{\frac{1}{2}\pi} e^x\sin x\,dx = (-e^x\cos x + e^x\sin x)\Big|_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} e^x\sin x\,dx,$$

and so

$$\int_0^{\frac{1}{2}\pi} e^x\sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x)\Big|_0^{\frac{1}{2}\pi} = \tfrac{1}{2}(e^{\frac{1}{2}\pi}+1).$$

Therefore (combining this with the previous result),

$$\int_0^{\frac{1}{2}\pi} e^x\sin^3 x\,dx = \tfrac{1}{10}e^{\frac{1}{2}\pi} + \tfrac{3}{10}(e^{\frac{1}{2}\pi}+1) = \tfrac{1}{10}(4e^{\frac{1}{2}\pi}+3).$$

Finally, applying the reduction formula to the second integral gives

$$\begin{aligned}\int_0^{\pi} e^{-x}\sin^4 x\,dx &= -\tfrac{1}{17}e^{-x}\sin^3 x(\sin x + 4\cos x)\Big|_0^{\pi} + \tfrac{12}{17}\int_0^{\pi} e^{-x}\sin^2 x\,dx\\ &= \tfrac{12}{17}\int_0^{\pi} e^{-x}\sin^2 x\,dx = -\tfrac{12}{85}e^{-x}\sin x(\sin x + 2\cos x)\Big|_0^{\pi} + \tfrac{24}{85}\int_0^{\pi} e^{-x}\,dx\\ &= \tfrac{24}{85}\int_0^{\pi} e^{-x}\,dx = \tfrac{24}{85}(1-e^{-\pi}).\end{aligned}$$

Solutions to the sample test

Solution to Question 1. a. Expanding the logarithm and then applying the product and chain rules gives

$$\frac{d}{dz}\left\{-\frac{\arcsin(z)}{z} - \log\left|\frac{1+\sqrt{1-z^2}}{z}\right|\right\} = \frac{\arcsin(z)}{z^2} - \frac{1}{z\sqrt{1-z^2}} - \frac{-z/\sqrt{1-z^2}}{1+\sqrt{1-z^2}} + \frac{1}{z}. \qquad (1)$$

When its denominator is rationalized the third term on the right becomes

$$-\frac{-z/\sqrt{1-z^2}}{1+\sqrt{1-z^2}} = \frac{z}{\sqrt{1-z^2}}\cdot\frac{1-\sqrt{1-z^2}}{z^2} = \frac{1-\sqrt{1-z^2}}{z\sqrt{1-z^2}},$$

and the second and fourth terms on the right combine to give

$$-\frac{1}{z\sqrt{1-z^2}} + \frac{1}{z} = \frac{-1+\sqrt{1-z^2}}{z\sqrt{1-z^2}}.$$

Since these last two displayed expressions sum to zero, the required equation follows. This equation (and the second fundamental theorem of calculus) implies that

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\arcsin(z)}{z^2}\,dz = \left\{-\frac{\arcsin(z)}{z} - \log\left|\frac{1+\sqrt{1-z^2}}{z}\right|\right\}\Bigg|_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}}$$

$$= -\tfrac{1}{3}\pi\cdot\tfrac{2}{3}\sqrt{3} + \tfrac{1}{6}\pi\cdot 2 - \log\left(\frac{1+\frac{1}{2}}{\frac{1}{2}\sqrt{3}}\cdot\frac{\frac{1}{2}}{1+\frac{1}{2}\sqrt{3}}\right)$$

$$= \tfrac{1}{9}(3-2\sqrt{3})\pi + \log\left(1+\tfrac{2}{3}\sqrt{3}\right).$$

b. i. If $\vartheta = \arctan(2/x)$, then $\tan(\vartheta) = 2/x$ and $\sec(\vartheta) = \sqrt{1+4/x^2}$. Therefore,

$$\sin\left(\arctan(2/x)\right) = \frac{\tan(\vartheta)}{\sec(\vartheta)} = \frac{2}{x\sqrt{1+4/x^2}} = \frac{2|x|}{x\sqrt{x^2+4}}.$$

ii. Since $x > 1$, $-\frac{1}{2}\pi < -1/x < 0$, and hence $\operatorname{arccot}\left(\cot(-1/x)\right) = \pi - 1/x$. Thus,

$$\arctan\left(\cot(-1/x)\right) = \tfrac{1}{2}\pi - \left(\pi - 1/x\right) = 1/x - \tfrac{1}{2}\pi.$$

iii. As $y \to 0^+$, $\log y \to -\infty$, and so $\operatorname{arcsec}(\log y) \to \frac{3}{2}\pi$.

Solution to Question 2. Each integral is computed using a standard integral and an implicit affine change of variables.

a. $\displaystyle\int \sqrt[4]{6x+7}\,dx = \tfrac{2}{15}(6x+7)^{5/4} + a$

b. $\displaystyle\int \frac{14}{\sqrt{(7w-3)^2+21}}\,dw = 2\ln|7w-3+\sqrt{(7w-3)^2+21}| + b$

c. $\displaystyle\int \cos(3y-5)\,dy = \tfrac{1}{3}\sin(3y-5) + c$

d. $\displaystyle\int \sec(2z+1)\,dz = \tfrac{1}{2}\log|\sec(2z+1)+\tan(2z+1)| + d$

Solution to Question 3. a. If $x = \sqrt{w-1}$, then $x^2 = w-1$, $2x\,dx = dw$ and $w+3 = x^2+4$. Also, $x = 2$ if $w = 5$, and $x = 2\sqrt{3}$ if $w = 13$. Therefore,

$$\int_5^{13} \frac{dw}{(w+3)\sqrt{w-1}} = 2\int_2^{2\sqrt{3}} \frac{dx}{x^2+4} = \arctan\left(\tfrac{1}{2}x\right)\Big|_2^{2\sqrt{3}} = \tfrac{1}{3}\pi - \tfrac{1}{4}\pi = \tfrac{1}{12}\pi.$$

b. If $w = \cos(ve^{-v})$, then

$$dw = -\sin(ve^{-v})(e^{-v} - ve^{-v})\,dv = \frac{(v-1)\sin(ve^{-v})}{e^v}\,dv.$$

Therefore,

$$\int \frac{(v-1)\sin(ve^{-v})}{e^v\cos(ve^{-v})}\,dv = \int \frac{dw}{w} = \log|\cos(ve^{-v})| + b.$$

c. If $y = e^x$ then $dy = e^x\,dx$, or $dy/y = dx$, and thus

$$\int \frac{dx}{e^x - 4e^{-x}} = \int \frac{dy}{y(y-4/y)} = \int \frac{dy}{y^2-4} = \tfrac{1}{4}\log\left|\frac{e^x-2}{e^x+2}\right| + c.$$

Solution to Question 4. a. Repeated partial integration, integrating the function of 4ϑ and then differentiating the function of 3ϑ, gives

$$\int \sin(3\vartheta)\cos(4\vartheta)\,d\vartheta = \tfrac{1}{4}\sin(3\vartheta)\sin(4\vartheta) - \tfrac{3}{4}\int \cos(3\vartheta)\sin(4\vartheta)d\vartheta$$

$$= \tfrac{1}{4}\sin(3\vartheta)\sin(4\vartheta) + \tfrac{3}{16}\cos(3\vartheta)\cos(4\vartheta) + \tfrac{9}{16}\int \sin(3\vartheta)\cos(4\vartheta)\,d\vartheta.$$

Absorbing the integral on the right then gives

$$\int \sin(3\vartheta)\cos(4\vartheta)\,d\vartheta = \tfrac{4}{7}\sin(3\vartheta)\sin(4\vartheta) + \tfrac{3}{7}\cos(3\vartheta)\cos(4\vartheta) + C.$$

b. Repeated partial integration, integrating the power and differentiating the (power of the) logarithm, yields

$$\int_1^e \left(\frac{\ln y}{y}\right)^2 dy = -\frac{(\ln y)^2}{y}\Bigg|_1^e + 2\int_1^e \frac{\ln y}{y^2}\,dy = -e^{-1} - \frac{2\ln y}{y}\Bigg|_1^e + \int_1^e \frac{dy}{y^2} = -3e^{-1} - \frac{2}{y}\Bigg|_1^e$$

$$= 2 - 5e^{-1}.$$

c. Repeated partial integration, integrating the exponential factor and then differentiating the polynomial, gives

$$\int (2t-1)^2 e^{-4t}\,dt = (2t-1)^2\cdot\frac{e^{-4t}}{-4} - 4(2t-1)\cdot\frac{e^{-4t}}{16} + 8\cdot\frac{e^{-4t}}{16} + k$$

$$= -\tfrac{1}{8}e^{-4t}\left(2(2t-1)^2 + 2(2t-1) + 1\right) + k$$

$$= -\tfrac{1}{8}e^{-4t}(8t^2 - 4t + 1) + k.$$

Solution to Question 5. a. If $y = e^{2x}+3$, then $\frac{1}{2}\,dy = e^{2x}\,dx$ and $e^{4x} = (y-3)^2$, so

$$\int e^{6x}\cos(e^{2x}+3)\,dx = \tfrac{1}{2}\int (y-3)^2\cos(y)\,dy$$

$$= \tfrac{1}{2}\left\{(y-3)^2\sin(y) - 2(y-3)(-\cos(y)) + 2(-\sin(y))\right\} + a$$

$$= \tfrac{1}{2}(e^{4x}-2)\sin(e^{2x}+3) + e^{2x}\cos(e^{2x}+3) + a.$$

b. If $y = 13-8x-5x^2$ then $dy = -(10x+8)\,dx$ and $3x+4 = \frac{10}{3}(10x+8) - \frac{68}{3}$. Also, $5(13-8x-5x^2) = 65-40x-25x^2 = 81-(5x+4)^2$, so

$$\int \frac{3x+4}{\sqrt{13-8x-5x^2}}\,dx = -\tfrac{10}{3}\int \frac{dy}{\sqrt{y}} + \tfrac{68}{3}\int \frac{dx}{\sqrt{13-8x-5x^2}}$$

$$= -\tfrac{20}{3}\sqrt{y} + \tfrac{68}{3}\sqrt{5}\int \frac{dx}{\sqrt{81-(5x+4)^2}}$$

$$= -\tfrac{20}{3}\sqrt{13-8x-5x^2} + \tfrac{68}{3}\sqrt{5}\arcsin\left(\frac{5x+4}{9}\right) + b.$$

c. If $y = \sqrt{1+\ln x}$, then $\ln x = y^2 - 1$, and $(dx)/x = 2y\,dy$. Hence,

$$\int \frac{\sqrt{1+\ln x}}{x(\ln x)^2}\,dx = \int \frac{2y^2}{(y^2-1)^2}\,dy.$$

Partial integration, integrating the right factor and differentiating y, gives

$$\int y \cdot \frac{2y}{(y^2-1)^2}\,dy = -\frac{y}{y^2-1} + \int \frac{dy}{y^2-1} = -\frac{y}{y^2-1} + \tfrac{1}{2}\ln\left|\frac{y-1}{y+1}\right| + c.$$

Therefore,

$$\int \frac{\sqrt{1+\ln x}}{x(\ln x)^2}\,dx = -\frac{\sqrt{1+\ln x}}{\ln x} + \tfrac{1}{2}\ln\left|\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right| + c.$$

d. Partial integration, integrating the power and differentiating the radical, gives

$$\int \frac{\sqrt{9x^2-5}}{x^2}\,dx = -\frac{\sqrt{9x^2-5}}{x} + 9\int \frac{dx}{\sqrt{9x^2-5}} = -\frac{\sqrt{9x^2-5}}{x} + 3\ln\left|3x+\sqrt{9x^2-5}\right| + d.$$

e. Partial integration, integrating the factor 1 and differentiating $\arctan(1/x)$, gives

$$\int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \arctan(1/x)\,dx = x\arctan(1/x)\Big|_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} - \int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{x}{1+1/x^2}\cdot\frac{-1}{x^2}\,dx$$

$$= \tfrac{1}{18}\pi\sqrt{3} + \int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{x}{1+x^2}\,dx.$$

By inspection,

$$\int_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\log(1+x^2)\Big|_{\frac{1}{3}\sqrt{3}}^{\sqrt{3}} = \tfrac{1}{2}\log 3.$$

Therefore, the integral in question is equal to $\frac{1}{18}\pi\sqrt{3} + \frac{1}{2}\log 3$.

f. Partial integration, integrating a constant and differentiating the logarithm, gives

$$\int_0^{\frac{1}{2}} \ln\left(\frac{1+x}{1-x}\right)dx = (1+x)\ln\left(\frac{1+x}{1-x}\right)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} (1+x)\cdot\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2}\,dx$$

$$= \tfrac{3}{2}\ln 3 - 2\int_0^{\frac{1}{2}} \frac{dx}{1-x} = \tfrac{3}{2}\ln 3 + 2\ln(1-x)\Big|_0^{\frac{1}{2}} = \tfrac{3}{2}\ln 3 + 2\ln\tfrac{1}{2}$$

$$= \tfrac{1}{2}\ln\tfrac{27}{16}.$$

Solution to Question 6. a. If $x+y=4$, then $dx=-dy$, $x-2=2-y$, $x^2-4x+8=y^2-4y+8$ and $\sin\left(\frac{1}{4}\pi x\right) = \sin\left(\pi-\frac{1}{4}\pi y\right) = \sin\left(\frac{1}{4}\pi y\right)$. Therefore, the integral is equal to

$$\int_1^3 \sin\left(\tfrac{1}{4}\pi y\right)\ln\left(2-y+\sqrt{y^2-4y+8}\right)dy,$$

an thus twice the integral is equal to

$$\int_1^3 \sin\left(\tfrac{1}{4}\pi x\right)\ln\left(\left(x-2+\sqrt{x^2-4x+8}\right)\left(2-x+\sqrt{x^2-4x+8}\right)\right)dx.$$

Since $\left(x-2+\sqrt{x^2-4x+8}\right)\left(2-x+\sqrt{x^2-4x+8}\right)=4$, it follows that twice the integral is equal to

$$2\log 2\int_1^3 \sin\left(\tfrac{1}{4}\pi x\right)dx = -\frac{8\log 2}{\pi}\cos\left(\tfrac{1}{4}\pi x\right)\Big|_1^3 = \frac{8(\log 2)\sqrt{2}}{\pi}.$$

Therefore,

$$\int_1^3 \sin\left(\tfrac{1}{4}\pi x\right)\ln\left(x-2+\sqrt{x^2-4x+8}\right)dx = 4\pi^{-1}(\log 2)\sqrt{2}.$$

b. Observe that

$$\frac{1}{(\alpha t^2+\beta)^n} = \frac{\alpha t^2+\beta}{(\alpha t^2+\beta)^{n+1}} = \frac{\alpha t^2}{(\alpha t^2+\beta)^{n+1}} + \frac{\beta}{(\alpha t^2+\beta)^{n+1}},$$

and hence,

$$\frac{\beta}{(\alpha t^2+\beta)^{n+1}} = -\frac{\alpha t^2}{(\alpha t^2+\beta)^{n+1}} + \frac{1}{(\alpha t^2+\beta)^n}. \qquad (2)$$

Integrating the first term on the right by parts gives

$$-\int t\cdot\frac{\alpha t}{(\alpha t^2+\beta)^{n+1}}\,dt = \frac{t}{2n(\alpha t^2+\beta)^n} - \frac{1}{2n}\int \frac{dt}{(\alpha t^2+\beta)^n}.$$

Integrating (2), using this last computation, and dividing by β then yields

$$\int \frac{dt}{(\alpha t^2+\beta)^{n+1}} = \frac{t}{2n\beta(\alpha t^2+\beta)^n} + \frac{2n-1}{2n\beta}\int \frac{dt}{(\alpha t^2+\beta)^n}.$$

To evaluate the first integral, apply the reduction formula twice. After two applications of the reduction formula the remaining integral vanishes (because the coefficient $2\cdot\frac{1}{2}-1$ is zero).

$$\begin{aligned}\int \frac{dx}{(x^2+2x+5)^{5/2}} &= \int \frac{dx}{((x+1)^2+4)^{5/2}}\\ &= \frac{x+1}{12((x+1)^2+4)^{3/2}} + \frac{1}{6}\int \frac{dx}{((x+1)^2+4)^{3/2}}\\ &= \frac{x+1}{12((x+1)^2+4)^{3/2}} + \frac{x+1}{24((x+1)^2+4)^{1/2}} + C\\ &= \frac{(x+1)(x^2+2x+7)}{(x^2+2x+5)^{5/2}} + C.\end{aligned}$$

To evaluate the second integral, apply the reduction formula three times, at which point the remaining integral is an inverse tangent function.

$$\begin{aligned}
\int \frac{dx}{(4x^2+9)^4} &= \frac{x}{54(4x^2+9)^3} + \frac{5}{54}\cdot\frac{x}{36(4x^2+9)^2} \\
&\quad + \frac{5}{54}\cdot\frac{1}{12}\cdot\frac{x}{18(4x^2+9)} + \frac{5}{54}\cdot\frac{1}{12}\cdot\frac{1}{18}\cdot\frac{1}{6}\arctan\tfrac{2}{3}x + C \\
&= \frac{x(80x^4+480x+891)}{2^4\cdot 3^6\cdot(4x^2+9)^3} + \frac{5}{2^5\cdot 3^7}\arctan\tfrac{2}{3}x + C \\
&= \frac{x(80x^4+480x+891)}{11664(4x^2+9)^3} + \tfrac{5}{69984}\arctan\tfrac{2}{3}x + C.
\end{aligned}$$
