Note. — You should also review the integrals and exercises on the first review sheet.

Exercise 1. — Evaluate each of the following integrals.

a. $\displaystyle\int_0^{\sqrt{\pi}} x\sin^3(x^2)\cos^4(x^2)\,dx$ b. $\displaystyle\int \frac{\csc^3(\log x)\cot^5(\log x)}{x}\,dx$ c. $\displaystyle\int \sin^6(2\vartheta)\,d\vartheta$

d. $\displaystyle\int \frac{dz}{z^2\sqrt{(5-z^2)^3}}$ e. $\displaystyle\int \sqrt{x}\sqrt{x+x\sqrt{x}}\,dx$ f. $\displaystyle\int x(\tan^{-1}x)^2\,dx$

g. $\displaystyle\int \frac{(4y^2-25)^{3/2}}{y^5}\,dy$ h. $\displaystyle\int \frac{\sqrt{x^2+4x+13}}{x+2}\,dx$ i. $\displaystyle\int \sqrt{3u^2-5u+4}\,du$

j. $\displaystyle\int_{-3}^{1} \frac{2x+5}{(x^2+4x+7)^2}\,dx$ k. $\displaystyle\int \frac{3x^3+6x-1}{x^4+x^2+1}\,dx$ l. $\displaystyle\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx$

m. $\displaystyle\int \frac{y+\sqrt{2y+3}}{(y+2)\sqrt{2y+3}}\,dy$ n. $\displaystyle\int x^2\sqrt{16x^2+9}\,dx$ o. $\displaystyle\int \frac{4x^4-9x^3+34x^2-29x+52}{(x+1)(x^2-2x+5)^2}\,dx$

p. $\displaystyle\int \frac{dy}{y^{4/3}(1+y^{2/3})^{3/2}}$ q. $\displaystyle\int \frac{x^2}{\sqrt{6x-x^2}}\,dx$ r. $\displaystyle\int \log(x^2-x+2)\,dx$

s. $\displaystyle\int \sec^3(\vartheta)\tan^2(\vartheta)\,d\vartheta$ t. $\displaystyle\int \frac{\sec^6(\sqrt{w})\tan^2(\sqrt{w})}{\sqrt{w}}\,dw$ u. $\displaystyle\int_0^{\frac{1}{6}\pi} \frac{d\vartheta}{2-\sin\vartheta}$

v. $\displaystyle\int \frac{dy}{e^{2y}+6e^{-y}-7}$ w. $\displaystyle\int \frac{x^2}{\sqrt[3]{(x+1)^2(x-2)^7}}\,dx$ x. $\displaystyle\int xe^x\cos(x)\,dx$

Exercise 2. — Evaluate each integral, or show that it diverges. Be specific. If an integral diverges to ∞ or to $-\infty$, then give this information as well, together with a brief justification.

a. $\displaystyle\int_0^1 \frac{(\log x)^2}{\sqrt{x}}\,dx$ b. $\displaystyle\int_1^\infty \frac{\sqrt{p-1}}{p^2\sqrt{p-1}}\,dp$ c. $\displaystyle\int_0^9 \frac{3y+2}{y^2-2y-8}\,dy$

d. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x\log(\sin x)}{\sin x}\,dx$ e. $\displaystyle\int_1^\infty \frac{\operatorname{arcsec} z}{z^3}\,dz$ f. $\displaystyle\int_1^\infty \frac{\sqrt{\operatorname{arccsc} x}}{\sqrt[3]{x}}\,dx$

g. $\displaystyle\int_{\sqrt{2}}^\infty \frac{dw}{w^6\sqrt{w^2-2}}$ h. $\displaystyle\int_1^\infty \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{x\sqrt{x^4-1}}\,dx$ i. $\displaystyle\int_0^\infty e^{-x}\cos(x)\,dx$

j. $\displaystyle\int_0^\infty \frac{\arctan x}{x\sqrt{x}}\,dx$ k. $\displaystyle\int_2^\infty \frac{4x+3}{(2x-1)(x+2)(x-1)}\,dx$ l. $\displaystyle\int_{-3}^{2} \frac{2x+1}{\sqrt[3]{(x^3-3x+2)^2}}\,dx$

Exercise 3. — Determine all values of r for which the improper integral

$$\int_0^\infty \frac{\sin x}{x^r}\,dx$$

is convergent.

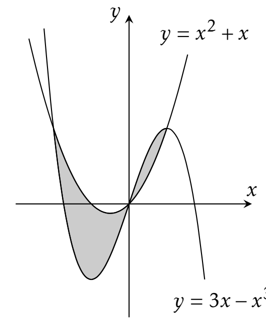
Exercise 4. — Compute the area of the region enclosed by the graphs of $y=x^2$ and $(x^2+1)y=2x$.

Exercise 5. — Express the area of the region enclosed by the graphs of

$$y=(x+1)(x^2-5x+6)(x^2+3x-10) \qquad\text{and}\qquad y=5(x-2)^2(x^2-1),$$

as the sum of definite integrals of polynomials. Use as few integrals as possible.

Exercise 6. — The region $\mathscr{R}$ enclosed by the graphs of $y=x^2+x$ and $y=3x-x^3$ is sketched below.



Let $\mathscr{R}_1$ denote the part of $\mathscr{R}$ to the left of the y-axis, and let $\mathscr{R}_2$ denote the part of $\mathscr{R}$ so the right of the y-axis. Write an integral which is equal to

a. the area of $\mathscr{R}$,
b. the volume of the solid obtained by revolving $\mathscr{R}_2$ about the line defined by $y=2$,
c. the volume of the solid obtained by revolving $\mathscr{R}_1$ about the line defined by $x=1$ and
d. the length of the perimeter of $\mathscr{R}$.

Evaluate the integrals in parts a and c.

Exercise 7. — The functions f and g are continuous on $[2,7]$, where $1\leqslant f(x)\leqslant g(x)\leqslant 4$, and $\mathscr{F}$ denotes the figure enclosed by their graphs and the lines defined by $x=2$ and $x=7$. If $\mathscr{F}$ is revolved about the line defined by $y=-1$, the volume of the resulting solid is 69π. If $\mathscr{F}$ is revolved about the line defined by $y=5$, the volume of the resulting solid is 39π. Determine the area of $\mathscr{R}$.

Exercise 8. — Find the length of each curve.

a. $x=\log(1-y^2)$, from $(0,0)$ to $\left(\log\frac{3}{4},\frac{1}{2}\right)$.
b. $y=\log x$, for $1\leqslant x\leqslant\sqrt{2}$.
c. $y=\log(\cos x)$, for $0\leqslant x\leqslant\frac{1}{3}\pi$.

Exercise 9. — Let $\mathscr{R}$ be the region enclosed by the graphs of

$$y=\sqrt{1-\cos x}, \qquad y=0, \qquad x=0 \qquad\text{and}\qquad x=\pi.$$

Compute the area of $\mathscr{R}$, and find the volume of the solid obtained by revolving $\mathscr{R}$ about the x-axis, the y-axis and the line defined by $y=2$.

Exercise 10. — The boundary and inside of a four dimensional sphere of radius r is the set of all points (x,y,z,t) such that $x^2+y^2+z^2+t^2\leqslant r^2$. Using the idea of integrating cross sections, compute the volume of a four dimensional sphere of radius r.

Solution to Exercise 1. — a. If $t=\cos(x^2)$, then $-\frac{1}{2}\,dt = x\sin(x^2)\,dx$. Also, $\sin^2(x^2)=1-t^2$, $t=1$ if $x=0$ and $t=-1$ if $x=\sqrt{\pi}$. Hence (using symmetry in the second equation)

$$\int_0^{\sqrt{\pi}} x\sin^3(x^2)\cos^4(x^2)\,dx = -\frac{1}{2}\int_1^{-1}(1-t^2)t^4\,dt = \int_0^1 (t^4-t^6)\,dt = \left(\tfrac{1}{5}t^5-\tfrac{1}{7}t^7\right)\Big|_0^1 = \frac{2}{35}.$$

b. If $t=\csc(\log x)$, then $-\,dt = x^{-1}\csc(\log x)\cot(\log x)\,dx$, and so

$$\int \frac{\csc^3(\log x)\cot^5(\log x)}{x}\,dx = -\int t^2(t^2-1)^2\,dt = -\int (t^6-2t^4+t^2)\,dt$$
$$= -\tfrac{1}{7}\csc^7(\log x)+\tfrac{2}{5}\csc^5(\log x)-\tfrac{1}{3}\csc^3(\log x)+C.$$

c. Applying the half-angle identity for the sine function gives

$$\int \sin^6(2\vartheta)\,d\vartheta = \frac{1}{8}\int (1-\cos(4\vartheta))^3\,d\vartheta = \frac{1}{8}\int\Big(1-3\cos(4\vartheta)+3\cos^2(4\vartheta)-\cos^3(4\vartheta)\Big)\,d\vartheta$$
$$= \tfrac{1}{8}\vartheta - \tfrac{3}{32}\sin(4\vartheta)+\tfrac{3}{8}\int\cos^2(4\vartheta)\,d\vartheta - \tfrac{1}{8}\int\cos^3(4\vartheta)\,d\vartheta.$$

Next, applying the half-angle identity for the cosine function to the first remaining integral gives

$$\int\cos^2(4\vartheta)\,d\vartheta = \frac{1}{2}\int(1+\cos(8\vartheta))\,d\vartheta = \tfrac{1}{2}\vartheta+\tfrac{1}{16}\sin(8\vartheta)+C,$$

and changing the variable of integration to $t=\sin 4\vartheta$ in the second remaining integral yields

$$\int\cos^3(4\vartheta)\,d\vartheta = \frac{1}{4}\int(1-t^2)\,dt = \tfrac{1}{4}t-\tfrac{1}{12}t^3+C = \tfrac{1}{4}\sin(4\vartheta)-\tfrac{1}{12}\sin^3(4\vartheta)+C.$$

Combining the foregoing results gives

$$\int\sin^6(2\vartheta)\,d\vartheta = \tfrac{5}{16}\vartheta-\tfrac{1}{8}\sin(4\vartheta)+\tfrac{1}{96}\sin^3(4\vartheta)+\tfrac{3}{128}\sin(8\vartheta)+C.$$

Note: Alternatively, one could use the reduction formula for powers of sine:

$$\int\sin^6(2\vartheta)\,d\vartheta = \tfrac{1}{2}\Big\{-\tfrac{1}{6}\sin^5(2\vartheta)\cos(2\vartheta)-\tfrac{5}{6}\tfrac{1}{4}\sin^3(2\vartheta)\cos(2\vartheta)-\tfrac{5}{6}\tfrac{3}{4}\tfrac{1}{2}\sin(2\vartheta)\cos(2\vartheta)-\tfrac{5}{6}\tfrac{3}{4}\tfrac{1}{2}(2\vartheta)\Big\}+C$$
$$= -\tfrac{1}{12}\sin^5(2\vartheta)\cos(2\vartheta)-\tfrac{5}{48}\sin^3(2\vartheta)\cos(2\vartheta)-\tfrac{5}{32}\sin(2\vartheta)\cos(2\vartheta)-\tfrac{5}{16}\vartheta+C.$$

d. If $y=z^{-1}\sqrt{5-z^2}$, then $y^2=5z^{-2}-1$, so $-\frac{1}{5}y\,dy = z^{-3}\,dz$ and $z^{-2}=\frac{1}{5}(y^2+1)$. Hence,

$$\int\frac{dz}{z^2\sqrt{(5-z^2)^3}} = \int\left(\frac{z}{\sqrt{5-z^2}}\right)^3\cdot\frac{1}{z^2}\cdot\frac{dz}{z^3} = \int\frac{1}{y^3}\cdot\frac{y^2+1}{5}\cdot\frac{-y\,dy}{5}$$
$$= -\frac{1}{25}\int\left(1+y^{-2}\right)dy = -\frac{1}{25}\left(y-y^{-1}\right)+C = \frac{1-y^2}{25y}+C$$
$$= \frac{(2-5z^{-2})z}{25\sqrt{5-z^2}}+C = \frac{2z^2-5}{25z\sqrt{5-z^2}}+C.$$

e. If $t=\sqrt{1+\sqrt{x}}$ then $x=(t^2-1)^2$ and $dx=4t(t^2-1)\,dt$; therefore,

$$\int\sqrt{x}\sqrt{x+x\sqrt{x}}\,dx = \int x\sqrt{1+\sqrt{x}}\,dx = 4\int t^2(t^2-1)^3\,dt = 4\int(t^8-3t^6+3t^4-t^2)\,dt$$
$$= 4\left\{\tfrac{1}{9}t^9-\tfrac{3}{7}t^7+\tfrac{3}{5}t^5-\tfrac{1}{3}t^3\right\}+C$$
$$= \tfrac{4}{315}t^3(35t^6-135t^4+189t^2-105)+C.$$

To express the result in terms of x in simplified form, observe that

$$t^6 = 1+3\sqrt{x}+3x+x\sqrt{x},$$
$$t^4 = 1+2\sqrt{x}+x \quad\text{and}$$
$$t^2 = 1+\sqrt{x},$$

and so

$$35t^6-135t^4+189t^2-105 = 35x\sqrt{x}-30x+24\sqrt{x}-16.$$

Therefore,

$$\int\sqrt{x}\sqrt{x+x\sqrt{x}}\,dx = \tfrac{4}{315}(35x\sqrt{x}-30x+24\sqrt{x}-16)\sqrt{(1+\sqrt{x})^3}+C.$$

f. As the notation $\tan^{-1}$ is repulsive, arctan will be written instead. Partial integration, taking the primitive $\frac{1}{2}(x^2+1)$ of x, gives

$$\int x(\arctan x)^2\,dx = \tfrac{1}{2}(x^2+1)(\arctan x)^2-\tfrac{1}{2}\int(x^2+1)\cdot\frac{2\arctan x}{x^2+1}\,dx$$
$$= \tfrac{1}{2}(x^2+1)(\arctan x)^2-\int\arctan x\,dx.$$

Now integrating by parts and by inspection gives

$$\int\arctan x\,dx = x\arctan x-\int\frac{x}{x^2+1}\,dx = x\arctan x-\tfrac{1}{2}\log(x^2+1)+C.$$

Therefore,

$$\int x(\arctan x)^2\,dx = \tfrac{1}{2}(x^2+1)(\arctan x)^2-x\arctan x+\tfrac{1}{2}\log(x^2+1)+C.$$

g. If $2y=5\sec\vartheta$, then $dy=\frac{5}{2}\sec\vartheta\tan\vartheta\,d\vartheta$, $\sqrt{4y^2-25}=5\tan\vartheta$, and hence

$$\int\frac{(4y^2-25)^{3/2}}{y^5}\,dy = \int\frac{(5\tan\vartheta)^3\left(\frac{5}{2}\sec\vartheta\tan\vartheta\right)}{\left(\frac{5}{2}\sec\vartheta\right)^5}\,d\vartheta = \frac{16}{5}\int\sin^4(\vartheta)\,d\vartheta.$$

The half angle identities for the sine and cosine functions give

$$\sin^4(\vartheta) = \tfrac{1}{4}\left(1-2\cos(2\vartheta)+\cos^2(2\vartheta)\right) = \tfrac{3}{8}-\tfrac{1}{2}\cos(2\vartheta)+\tfrac{1}{8}\cos(4\vartheta).$$

Therefore,

$$\frac{16}{5}\int\sin^4(\vartheta)\,d\vartheta = \tfrac{6}{5}\vartheta-\tfrac{4}{5}\sin(2\vartheta)+\tfrac{1}{10}\sin(4\vartheta)+C.$$

The double angle identities for the sine and cosine function imply that

$$-\tfrac{4}{5}\sin(2\vartheta)+\tfrac{1}{10}\sin(4\vartheta) = \tfrac{1}{5}(\cos(2\vartheta)-4)\sin(2\vartheta) = \tfrac{2}{5}\left(2\cos^2(\vartheta)-5\right)\sin\vartheta\cos\vartheta.$$

Now $\vartheta=\operatorname{arcsec}\left(\frac{2}{5}y\right)$, $\sin\vartheta = \dfrac{\sqrt{4y^2-25}}{2y}$, $\cos\vartheta=\dfrac{5}{2y}$ and $2\cos^2\vartheta-5 = 2\left(\dfrac{5}{2y}\right)^2-5 = \dfrac{5(5-2y^2)}{2y^2}$; hence,

$$\int\frac{(4y^2-25)^{3/2}}{y^5}\,dy = \tfrac{6}{5}\operatorname{arcsec}\left(\tfrac{2}{5}y\right)+\frac{5(5-2y^2)\sqrt{4y^2-25}}{4y^4}+C.$$

Note. — The integral in question could be evaluated by repeated partial integration. Integrating the power of y and differentiating the radical expression at each stage gives

$$\begin{aligned}\int \frac{(4y^2-25)^{3/2}}{y^5}\,dy &= -\frac{(4y^2-25)^{3/2}}{4y^4} + 3\int \frac{\sqrt{4y^2-25}}{y^3}\,dy \\ &= -\frac{(4y^2-25)^{3/2}}{4y^4} - \frac{3\sqrt{4y^2-25}}{2y^2} + 6\int \frac{2\,dy}{2y\sqrt{4y^2-25}} \\ &= -\frac{5(2y^2-5)\sqrt{4y^2-25}}{4y^4} + \tfrac{6}{5}\operatorname{arcsec}\left(\tfrac{2}{5}y\right) + C.\end{aligned}$$

h. If $y = \sqrt{x^2+4x+13} = \sqrt{(x+2)^2+9}$ then $(x+2)^2 = y^2-9$ and $(x+2)\,dx = y\,dy$, so

$$\begin{aligned}\int \frac{\sqrt{x^2+4x+13}}{x+2}\,dx &= \int \frac{y^2}{y^2-9}\,dy = \int \left(1 + \frac{9}{y^2-9}\right)dy = y + \tfrac{3}{2}\log\left|\frac{y-3}{y+3}\right| + C \\ &= \sqrt{x^2+4x+13} + \tfrac{3}{2}\log\left|\frac{\sqrt{x^2+4x+13}-3}{\sqrt{x^2+4x+13}+3}\right| + C.\end{aligned}$$

i. Partial integration, using the primitive $\frac{1}{6}(6u-5)$ of u, and noting $12(3u^2-5u+4) = (6u-5)^2+23$, gives

$$\begin{aligned}\int \sqrt{3u^2-5u+4}\,du &= \tfrac{1}{6}(6u-5)\sqrt{3u^2-5u+4} - \tfrac{1}{12}\int \frac{(6u-5)^2+23-23}{\sqrt{3u^2-5u+4}}\,du \\ &= \tfrac{1}{6}(6u-5)\sqrt{3u^2-5u+4} - \int \sqrt{3u^2-5u+4}\,du + \frac{23}{2\sqrt{3}}\int \frac{du}{\sqrt{(6u-5)^2+23}} \\ &= \tfrac{1}{12}(6u-5)\sqrt{3u^2-5u+4} + \tfrac{23}{72}\sqrt{3}\log\left(6u-5+2\sqrt{3}\sqrt{3u^2-5u+4}\right) + C,\end{aligned}$$

where the second term on the second line is absorbed by the left side.

j. If $t = x+2$ then

$$\int_{-3}^{1} \frac{2x+5}{(x^2+4x+7)^2}\,dx = \int_{-1}^{3} \frac{2t+1}{(t^2+3)^2}\,dt = -\frac{1}{t^2+3}\bigg|_{-1}^{3} + \int_{-1}^{3} \frac{dt}{(t^2+3)^2} = \tfrac{1}{6} + \int_{-1}^{3} \frac{dt}{(t^2+3)^2}.$$

Adding and subtracting t^2 in the numerator and then integrating by parts gives

$$\begin{aligned}\int_{-1}^{3} \frac{3\,dt}{(t^2+3)^2} &= \int_{-1}^{3} \frac{dt}{t^2+3} - \int_{-1}^{3} t\cdot\frac{t}{(t^2+3)^2}\,dt = \int_{-1}^{3} \frac{dt}{t^2+3} + \frac{t}{2(t^2+3)}\bigg|_{-1}^{3} - \int_{-1}^{3} \frac{dt}{2(t^2+3)} \\ &= \tfrac{1}{4} + \tfrac{1}{6}\sqrt{3}\arctan\left(\tfrac{1}{3}t\sqrt{3}\right)\bigg|_{-1}^{3} = \tfrac{1}{4} + \tfrac{1}{12}\pi\sqrt{3}.\end{aligned}$$

Therefore, the integral in question is equal to $\frac{1}{6} + \frac{1}{3}\left(\frac{1}{4} + \frac{1}{12}\pi\sqrt{3}\right) = \frac{1}{4} + \frac{1}{36}\pi\sqrt{3}$.

k. Since $x^4+x^2+1 = (x^2-x+1)(x^2+x+1)$, there are rational numbers a, b, c and d such that

$$\frac{ax+b}{x^2-x+1} + \frac{cx+d}{x^2+x+1} = \frac{3x^3+6x-1}{x^4+x^2+1}.$$

Clearing denominators gives

$$(ax+b)(x^2+x+1) + (cx+d)(x^2-x+1) = 3x^3+6x-1,$$

and comparing the cubic, quadratic, linear and constant coefficients gives, respectively,

$$a+c=3, \qquad a+b-c+d=0, \qquad a+b+c-d=6 \qquad \text{and} \qquad b+d=-1.$$

Subtracting the first equation from the second and third equations gives

$$b-2c+d=-3 \qquad \text{and} \qquad b-d=3.$$

Subtracting the fourth equation from the fifth and sixth equations gives $-2c=-2$, or $c=1$, and $-2d=4$, or $d=-2$. The first and fourth equations then give $a=2$ and $b=1$. Now $2x+1 = 2x-1+2$ and $4(x^2-x+1) = (2x-1)^2+3$, so

$$\int \frac{2x+1}{x^2-x+1}\,dx = \int \frac{2x-1}{x^2-x+1}\,dx + \int \frac{8\,dx}{(2x-1)^2+3} = \log(x^2-x+1) + \tfrac{4}{3}\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

Also, $x-2 = \frac{1}{2}(2x+1) - \frac{5}{2}$ and $4(x^2+x+1) = (2x+1)^2+3$, so

$$\int \frac{x-2}{x^2+x+1}\,dx = \tfrac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \int \frac{10\,dx}{(2x+1)^2+3} = \tfrac{1}{2}\log(x^2+x+1) - \tfrac{5}{3}\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

Therefore, the integral in question is equal to

$$\log\left((x^2-x+1)\sqrt{x^2+x+1}\right) + \tfrac{4}{3}\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) - \tfrac{5}{3}\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

l. Division gives

$$\frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1} = x+1 + \frac{-2x+4}{x^4-2x^3+2x^2-2x+1},$$

and $\frac{1}{2}x^2+x$ is a primitive of the quotient. Next, 1 is a zero of multiplicity two of the denominator, and factorizing by inspection gives $x^4-2x^3+2x^2-2x+1 = (x-1)^2(x^2+1)$. The resolution into partial fractions of the proper part has the form

$$\frac{-2x+4}{(x-1)^2(x^2+1)} = \frac{a}{x-1} + \frac{1}{(x-1)^2} + \frac{bx+c}{x^2+1},$$

in which the coefficient over $(x-1)^2$ is obtained by inspection (covering and evaluating). Clearing denominators gives

$$-2x+4 = a(x-1)(x^2+1) + (x^2+1) + (bx+c)(x^2-2x+1),$$

and comparing the cubic, quadratic and linear coefficients gives, respectively,

$$a+b=0, \qquad -a-2b+c=-1 \qquad \text{and} \qquad a+b-2c=-2.$$

Using the first equation to eliminate a from the second and third equations yields $-b+c=-1$ and $-2c=-2$, so $c=1$, $b=2$ and $a=-2$. Therefore, the integral of the proper part is

$$\int \left\{\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}\right\}dx = -2\log|x-1| - \frac{1}{x-1} + \log(x^2+1) + \arctan x + C.$$

Therefore,

$$\int \frac{x^5-x^4-3x+5}{x^4-2x^3+2x^2-2x+1}\,dx = \tfrac{1}{2}x^2 + x - \frac{1}{x-1} + \log\frac{x^2+1}{(x-1)^2} + \arctan x + C.$$

m. If $t = \sqrt{2y+3}$ then $y = \frac{1}{2}(t^2-3)$, $dy = t\,dt$ and $y+2 = \frac{1}{2}(t^2+1)$. Hence,

$$\begin{aligned}\int \frac{y+\sqrt{2y+3}}{(y+2)\sqrt{2y+3}}\,dy &= \int \frac{\frac{1}{2}(t^2-3)+t}{\frac{1}{2}(t^2+1)t}\cdot t\,dt = \int \frac{t^2+2t-3}{t^2+1}\,dt \\ &= \int \left\{1 + \frac{2t-4}{t^2+1}\right\}dt = t + \log(t^2+1) - 4\arctan(t) + C \\ &= \sqrt{2y+3} + \log(y+2) - 4\arctan(\sqrt{2y+3}) + C.\end{aligned}$$

n. If $y = \sqrt{16x^2+9}$ then $16x^2 = y^2 - 9$ and $y\,dy = 16x\,dx$. So

$$\begin{aligned} d(16x^3y) &= 48x^2y\,dx + 16x^3\,dy = 48x^2y\,dx + xy^2\,dy - 9x\,dy \\ &= 64x^2y\,dx - 9x\,dy, \end{aligned}$$

and

$$\begin{aligned} d(xy) &= y\,dx + x\,dy = \frac{9}{y}\,dx + \frac{16x^2}{y}\,dx + x\,dy \\ &= \frac{9}{y}\,dx + 2x\,dy. \end{aligned}$$

Therefore, the integral in question is equal to

$$\begin{aligned} \int x^2y\,dx &= \tfrac{1}{4}x^3y + \tfrac{9}{64}\int x\,dy = \tfrac{1}{4}x^3y + \tfrac{9}{128}\left(xy - 9\int\frac{dx}{y}\right) \\ &= \tfrac{1}{128}xy(32x^2+9) - \tfrac{81}{512}\log(4x+y) + C \\ &= \tfrac{1}{128}x(32x^2+9)\sqrt{16x^2+9} - \tfrac{81}{512}\log\Big(4x+\sqrt{16x^2+9}\Big) + C. \end{aligned}$$

o. The resolution into partial fractions of the integrand has the form

$$\frac{4x^4-9x^3+34x^2-29x+52}{(x+1)(x^2-2x+5)^2} = \frac{a}{x+1} + \frac{bx+c}{x^2-2x+5} + \frac{Dx+E}{(x^2-2x+5)^2},$$

in which $a = 2$ by inspection (covering and evaluating). Clearing denominators yields

$$4x^4-9x^3+34x^2-29x+52 = a(x^2-2x+5)^2 + (bx+c)(x+1)(x^2-2x+5) + (dx+e)(x+1)$$

and then comparing the coefficients of powers of x in decreasing order gives: $a+b=4$, so $b=2$, $-4a-b+c=-9$, so $c=1$, $14a+3b-c+d=34$, so $D=1$ and $-20A+5B+3C+D+E=-29$, so $E=-3$. Therefore, the integral is equal to

$$\int\left\{\frac{2}{x+1} + \frac{2x+1}{x^2-2x+5} + \frac{x-3}{(x^2-2x+5)^2}\right\}dx.$$

A primitive of the first term is $\log(x+1)^2$. Next,

$$\int\frac{2x+1}{x^2-2x+5}\,dx = \int\frac{2x-2}{x^2-2x+5}\,dx + \int\frac{3\,dx}{x^2-2x+5} = \log(x^2-2x+4) + \tfrac{3}{2}\arctan\Big(\tfrac{1}{2}(x-2)\Big) + C,$$

and

$$\int\frac{x-3}{(x^2-2x+5)^2}\,dx = \int\frac{x-1}{(x^2-2x+5)^2}\,dx - \int\frac{2\,dx}{(x^2-2x+5)^2} = \frac{-1}{2(x^2-2x+5)} - \int\frac{2\,dx}{(x^2-2x+5)^2}.$$

Adding and subtracting t^2 in the numerator and integrating by parts gives

$$\int\frac{4\,dt}{(t^2+4)^2} = \int\frac{dt}{t^2+4} - \int t\cdot\frac{t}{(t^2+4)^2}\,dt = \frac{t}{2(t^2+4)} + \frac{1}{2}\int\frac{dt}{t^2+4}.$$

Thus, with $t = x-1$ this last calculation shows that

$$\int\frac{2\,dx}{(x^2-2x+5)^2} = \frac{x-1}{4(x^2-2x+5)} + \tfrac{1}{8}\arctan\Big(\tfrac{1}{2}(x-1)\Big) + C.$$

Combining these results gives,

$$\begin{aligned} &\int\frac{4x^4-9x^3+34x^2-29x+52}{(x+1)(x^2-2x+5)^2}\,dx \\ &\quad = -\frac{x+1}{4(x^2-2x+5)} + \log\Big((x+1)^2(x^2-2x+5)\Big) + \tfrac{11}{8}\arctan\Big(\tfrac{1}{2}(x-1)\Big) + C. \end{aligned}$$

p. If $t = y^{-1/3}\sqrt{(1+y^{2/3})}$, then

$$t^2 - 1 = y^{-2/3}, \qquad -3t\,dt = y^{-5/3}\,dy \qquad \text{and} \qquad t^{-3}(t^2-1) = y^{1/3}(1+y^{2/3})^{-3/2}.$$

Therefore,

$$\begin{aligned} \int y^{-4/3}(1+y^{2/3})^{-3/2}\,dy &= -3\int t^{-2}(t^2-1)\,dt = -3\int(1-t^{-2})\,dt = -3(t+t^{-1}) + C \\ &= -\frac{3(t^2+1)}{t} + C = -\frac{3(y^{-2/3}+2)}{y^{-1/3}\sqrt{(1+y^{2/3})}} + C \\ &= -\frac{3(1+2y^{2/3})}{y^{1/3}\sqrt{(1+y^{2/3})}} + C. \end{aligned}$$

q. If $t = x-3$, then $\sqrt{6x-x^2} = \sqrt{9-t^2}$, $x^2 = (t+3)^2 = t^2+6t+9$. Partial integration gives

$$\begin{aligned} \int\frac{t^2}{\sqrt{9-t^2}}\,dt &= -t\sqrt{9-t^2} + \int\sqrt{9-t^2}\,dt = -t\sqrt{9-t^2} + 9\int\frac{dt}{\sqrt{9-t^2}} - \int\frac{t^2}{\sqrt{9-t^2}}\,dt \\ &= -\tfrac{1}{2}t\sqrt{9-t^2} + \tfrac{9}{2}\int\frac{dt}{\sqrt{9-t^2}}, \end{aligned}$$

and by inspection,

$$\int\frac{6t}{\sqrt{9-t^2}}\,dt = -\sqrt{9-t^2} + C.$$

Therefore,

$$\begin{aligned} \int\frac{x^2}{\sqrt{6x-x^2}}\,dx &= -\tfrac{1}{2}(t+12)\sqrt{9-t^2} + \tfrac{27}{2}\int\frac{dt}{\sqrt{9-t^2}} \\ &= -\tfrac{1}{2}(x+9)\sqrt{6x-x^2} + \tfrac{27}{2}\arcsin\Big(\tfrac{1}{3}(x-3)\Big) + C. \end{aligned}$$

Note. — It is also possible to use the change of variables $x-3 = 3\sin\vartheta$. In this case, $dx = 3\cos\vartheta\,d\vartheta$ and $\sqrt{6x-x^2} = 3\cos\vartheta$. Hence,

$$\begin{aligned} \int\frac{x^2}{\sqrt{6x-x^2}}\,dx &= \int\frac{\big(3(1+\sin\vartheta)\big)^2}{3\cos\vartheta}\cdot 3\cos\vartheta\,d\vartheta = 9\int(1+2\sin\vartheta+\sin^2\vartheta)\,d\vartheta \\ &= 9\vartheta - 18\cos\vartheta + \tfrac{9}{2}\int(1-\cos 2\vartheta)\,d\vartheta \\ &= \tfrac{27}{2}\vartheta - 18\cos\vartheta - \tfrac{9}{4}\sin 2\vartheta + C \\ &= \tfrac{27}{2}\vartheta - 18\cos\vartheta - \tfrac{9}{2}\sin\vartheta\cos\vartheta + C. \end{aligned}$$

Now

$$\cos\vartheta = \tfrac{1}{3}\sqrt{6x-x^2}, \qquad \text{and} \qquad \sin\vartheta\cos\vartheta = \tfrac{1}{9}(x-3)\sqrt{6x-x^2},$$

and so

$$\begin{aligned} \int\frac{x^2}{\sqrt{6x-x^2}}\,dx &= \tfrac{27}{2}\arcsin\Big(\tfrac{1}{3}(x-3)\Big) - 6\sqrt{6x-x^2} - \tfrac{1}{2}(x-3)\sqrt{6x-x^2} + C \\ &= \tfrac{27}{2}\arcsin\Big(\tfrac{1}{3}(x-3)\Big) - \tfrac{1}{2}(x+9)\sqrt{6x-x^2} + C. \end{aligned}$$

r. Integrating by parts gives

$$\begin{aligned} \int\log(x^2-x+2)\,dx &= \tfrac{1}{2}(2x-1)\log(x^2-x+2) - \tfrac{1}{2}\int\frac{(2x-1)^2}{x^2-x+2}\,dx \\ &= \tfrac{1}{2}(2x-1)\log(x^2-x+2) - 2\int\left\{1 - \frac{7}{(2x-1)^2+7}\right\}dx \\ &= \tfrac{1}{2}(2x-1)\log(x^2-x+2) - 2x + \sqrt{7}\arctan\Big(\tfrac{1}{7}(2x-1)\sqrt{7}\Big) + C. \end{aligned}$$

s. Since

$$\frac{d}{dx}\Big\{\sec^3(\vartheta)\tan(\vartheta)\Big\} = 3\sec^3(\vartheta)\tan^2(\vartheta)+\sec^5(\vartheta) = 4\sec^3(\vartheta)\tan^2(\vartheta)+\sec^3(\vartheta),$$

and

$$\frac{d}{dx}\Big\{\sec(\vartheta)\tan(\vartheta)\Big\} = \sec(\vartheta)\tan^2(\vartheta)+\sec^3(\vartheta) = -\sec(\vartheta)+2\sec^3(\vartheta),$$

it follows that

$$\frac{d}{dx}\Big\{2\sec^3(\vartheta)\tan(\vartheta)-\sec(\vartheta)\tan(\vartheta)\Big\} = 8\sec^3(\vartheta)\tan^2(\vartheta)+\sec(\vartheta).$$

Rearranging this last equation gives

$$\int \sec^3(\vartheta)\tan^2(\vartheta)\,d\vartheta = \tfrac{1}{4}\sec^3(\vartheta)\tan(\vartheta)-\tfrac{1}{8}\sec(\vartheta)\tan(\vartheta)-\tfrac{1}{8}\log|\sec(\vartheta)+\tan(\vartheta)|+C.$$

t. If $t=\tan(\sqrt{w})$, then $2\,dt = w^{-1/2}\sec^2(\sqrt{w})\,dw$ and $\sec^4(\sqrt{w}) = (t^2+1)^2 = t^4+2t^2+1$. Hence,

$$\begin{aligned}\int \frac{\sec^6(\sqrt{w})\tan^2(\sqrt{w})}{\sqrt{w}}\,dw &= 2\int t^2(t^4+2t^2+1)\,dt = 2\int (t^6+2t^4+t^2)\,dt\\ &= \tfrac{2}{7}\tan^7(\sqrt{w})+\tfrac{4}{5}\tan^5(\sqrt{w})+\tfrac{2}{3}\tan^3(\sqrt{w})+C.\end{aligned}$$

u. If $z=\tan\left(\tfrac{1}{2}\vartheta\right)$, then

$$\sin\vartheta = \frac{2z}{1+z^2} \qquad\text{and}\qquad dz = \frac{2\,dz}{1+z^2}.$$

If $\vartheta = 0$ then $z=0$, and if $\vartheta = \tfrac{1}{6}\pi$ then

$$z = \tan\left(\tfrac{1}{12}\pi\right) = \frac{\sin\left(\tfrac{1}{6}\pi\right)}{1+\cos\left(\tfrac{1}{6}\pi\right)} = \frac{1}{2+\sqrt{3}} = 2-\sqrt{3}.$$

The inverse tangent is an odd function, so the last calculation implies that $\arctan(\sqrt{3}-2) = -\tfrac{1}{12}\pi$. Therefore,

$$\begin{aligned}\int_0^{\frac{1}{6}\pi}\frac{d\vartheta}{2-\sin\vartheta} &= \int_0^{2-\sqrt{3}}\frac{1}{2-\frac{2z}{1+z^2}}\cdot\frac{2\,dz}{1+z^2} = \int_0^{2-\sqrt{3}}\frac{4\,dz}{(2z-1)^2+3} = \tfrac{2}{3}\sqrt{3}\arctan\left(\frac{2z-1}{\sqrt{3}}\right)\Bigg|_0^{2-\sqrt{3}}\\ &= \tfrac{2}{3}\sqrt{3}\left\{\arctan(\sqrt{3}-2)-\arctan\left(-\tfrac{1}{3}\sqrt{3}\right)\right\}\\ &= \tfrac{2}{3}\sqrt{3}\left(-\tfrac{1}{12}\pi+\tfrac{1}{6}\pi\right)\\ &= \tfrac{1}{18}\pi\sqrt{3}.\end{aligned}$$

v. If $z=e^y$ then $\log z = y$ and $z^{-1}\,dz = dy$, so

$$\int\frac{dy}{e^{2y}+6e^{-y}-7} = \int\frac{dz}{z^3-7z+6}.$$

The denominator of the integrand vanishes if $z=1$, so it is divisible by $z-1$, and factorizing by inspection gives $z^3-7z+6 = (z-1)(z^2+z-6) = (z-1)(z-2)(z+3)$. Resolving the integrand into partial fractions by inspection (*i.e.*, covering and evaluating) and then integrating gives

$$\int\frac{dz}{z^3-7z+6} = \int\left\{-\frac{1}{4(z-1)}+\frac{1}{5(z-2)}+\frac{1}{20(z+3)}\right\}dz = \tfrac{1}{20}\log\left|\frac{(z-2)^4(z+3)}{(z-1)^5}\right|+C;$$

therefore,

$$\int\frac{dy}{e^{2y}+6e^{-y}-7} = \tfrac{1}{20}\log\left|\frac{(e^y-2)^4(e^y+3)}{(e^y-1)^5}\right|+C.$$

w. By elementary properties of exponents,

$$\int\frac{x^2}{\sqrt[3]{(x+1)^2(x-2)^7}}\,dx = \int\frac{x^2}{(x+1)(x-2)^2}\sqrt[3]{\frac{x+1}{x-2}}\,dx.$$

Let t be the radical factor of the last integrand, so that

$$t^3 = \frac{x+1}{x-2},\quad -t^2\,dt = \frac{dx}{(x-2)^2},\quad x = \frac{2t^3+1}{t^3-1},\quad x+1 = \frac{3t^3}{t^3-1}\quad\text{and}\quad x^2 = \frac{4t^6+4t^3+1}{(t^3-1)^2}.$$

Therefore,

$$\int\frac{x^2}{\sqrt[3]{(x+1)^2(x-2)^7}}\,dx = \int\frac{t^3-1}{3t^3}\cdot\frac{4t^6+4t^3+1}{(t^3-1)^2}\cdot t\cdot(-t^2)\,dt = -\tfrac{1}{3}\int\frac{4t^6+4t^3+1}{t^3-1}\,dt.$$

Division and resolution into partial fractions gives

$$\frac{4t^6+4t^3+1}{t^3-1} = 4t^3+8+\frac{9}{(t-1)(t^2+t+1)} = 4t^3+8+\frac{3}{t-1}-\frac{3(t+2)}{t^2+t+1}. \tag{1}$$

Now

$$\int\frac{t+2}{t^2+t+1}\,dt = \tfrac{1}{2}\int\frac{2t+1}{t^2+t+1}\,dt+6\int\frac{dt}{(2t+1)^2+3} = \tfrac{1}{2}\log(t^2+t+1)+\sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)+C.$$

Integrating the remaining terms of (1) by inspection and then combining the results gives

$$\int\frac{x^2}{\sqrt[3]{(x+1)^2(x-2)^7}}\,dx = -\tfrac{1}{3}(t^4+8t)+\tfrac{1}{2}\log\left|\frac{t^2+t+1}{(t-1)^2}\right|+\sqrt{3}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)+C.$$

The polynomial function of t is

$$-\tfrac{1}{3}(t^3+8)t = -\frac{9x-15}{3(x-2)}\sqrt[3]{\frac{x+1}{x-2}} = -\frac{(3x-5)\sqrt[3]{x+1}}{(x-2)^{4/3}}.$$

The argument of the logarithm is the absolute value of

$$\frac{t^3-1}{(t-1)^3} = \frac{3}{x-2}\left(\sqrt[3]{\frac{x+1}{x-2}}-1\right)^{-3} = \frac{3}{\left(\sqrt[3]{x+1}-\sqrt[3]{x-2}\right)^3},$$

so the logarithmic term of the integral is $-\tfrac{3}{2}\log\left|\sqrt[3]{x+1}-\sqrt[3]{x-2}\right|$. The numerator of the argument of the inverse tangent is

$$2t+1 = 2\sqrt[3]{\frac{x+1}{x-2}}+1 = \frac{2\sqrt[3]{x+1}+\sqrt[3]{x-2}}{\sqrt[3]{x-2}}.$$

Therefore,

$$\begin{aligned}&\int\frac{x^2}{\sqrt[3]{(x+1)^2(x-2)^7}}\,dx\\ &\qquad = -\frac{(3x-5)\sqrt[3]{x+1}}{(x-2)^{4/3}}-\tfrac{3}{2}\log\left|\sqrt[3]{x+1}-\sqrt[3]{x-2}\right|+\sqrt{3}\arctan\left(\frac{2\sqrt[3]{x+1}+\sqrt[3]{x-2}}{\sqrt{3}\sqrt[3]{x-2}}\right)+C.\end{aligned}$$

x. Since

$$\frac{d}{dx}\Big\{xe^x\cos(x)\Big\} = e^x\cos(x)+xe^x\cos(x)-xe^x\sin(x)$$

and

$$\frac{d}{dx}\Big\{xe^x\sin(x)\Big\} = e^x\sin(x)+xe^x\cos(x)+xe^x\sin(x),$$

it follows that

$$\frac{d}{dx}\Big\{xe^x(\cos(x)+\sin(x))\Big\} = 2xe^x\cos(x)+e^x(\cos(x)+\sin(x)) = 2xe^x\cos(x)+\frac{d}{dx}\Big\{e^x\sin(x)\Big\}.$$

Therefore,

$$\int xe^x\cos(x)\,dx = \tfrac{1}{2}e^x\Big(x\big(\cos(x)+\sin(x)\big)-\sin(x)\Big)+C.$$

Solution to Exercise 2. — a. Partial integration gives

$$\int \frac{(\log x)^2}{\sqrt{x}}\,dx = 2(\log x)^2\sqrt{x} - 4\int\frac{\log x}{\sqrt{x}}\,dx = 2(\log x)^2\sqrt{x} - 8(\log x)\sqrt{x} + 8\int\frac{dx}{\sqrt{x}}$$
$$= 2(\log x)^2\sqrt{x} - 8(\log x)\sqrt{x} + 16\sqrt{x} + C.$$

Since $\varepsilon^p(\log\varepsilon)^n \to 0$ as $\varepsilon\to 0^+$ if $p>0$ and n is a positive integer, it follows that

$$\int_0^1 \frac{(\log x)^2}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1\frac{(\log x)^2}{\sqrt{x}}\,dx$$
$$= 16 - \lim_{\varepsilon\to 0^+}\Big\{2(\log\varepsilon)^2\sqrt{\varepsilon} - 8(\log\varepsilon)\sqrt{\varepsilon} + 16\sqrt{\varepsilon}\Big\}$$
$$= 16.$$

b. Changing the variable of integration to $\vartheta = \operatorname{arcsec}(\sqrt{p})$, so that $p=\sec^2\vartheta$, gives

$$dp = 2\sec^2\vartheta\tan\vartheta\,d\vartheta, \qquad \sqrt{p}-1 = \sec\vartheta - 1 \qquad \text{and} \qquad \sqrt{p-1} = \tan\vartheta.$$

Also, $\vartheta\to 0^+$ as $p\to 1^+$ and $\vartheta \to \frac{1}{2}\pi^-$ as $p\to\infty$. Therefore,

$$\int_1^\infty \frac{\sqrt{p}-1}{p^2\sqrt{p-1}}\,dp = \int_0^{\frac{1}{2}\pi}\frac{(\sec\vartheta-1)(2\sec^2\vartheta\tan\vartheta)}{\sec^4\vartheta\tan\vartheta}\,d\vartheta = 2\int_0^{\frac{1}{2}\pi}(\cos\vartheta-\cos^2\vartheta)\,d\vartheta.$$

The discontinuities of the integrand are removable, so the resulting integral is (essentially) definite. Therefore,

$$\int_1^\infty \frac{\sqrt{p}-1}{p^2\sqrt{p-1}}\,dp = 2\int_0^{\frac{1}{2}\pi}\cos\vartheta\,d\vartheta - 2\int_0^{\frac{1}{2}\pi}\cos^2\vartheta\,d\vartheta = (2\sin\vartheta)\Big|_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi}(1+\cos 2\vartheta)\,d\vartheta$$
$$= 2 - \Big\{\vartheta + \tfrac{1}{2}\sin 2\vartheta\Big\}\Big|_0^{\frac{1}{2}\pi}$$
$$= 2 - \tfrac{1}{2}\pi.$$

c. If $4<y\leqslant 5$, then $3y+2>14$ and $y+2\leqslant 7$, so

$$\frac{3y+2}{y^2-2y-8} = \frac{3y+2}{(y+2)(y-4)} > \frac{2}{y-4} > \frac{1}{y-4} > 0, \qquad \text{and} \qquad \int_4^5\frac{dy}{y-4} = \int_0^1\frac{dx}{x}$$

is divergent ($p=1$ in the scale of powers at the origin), so the improper integral

$$\int_0^9 \frac{3y+2}{y^2-2y-8}\,dy$$

is divergent, by the comparison principle.

d. Since $\frac{d}{dx}\big\{\log(\sin x)\big\} = \frac{\cos x}{\sin x}$, integrating by inspection gives

$$\int_0^{\frac{1}{2}\pi}\frac{\cos x\log(\sin x)}{\sin x}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^{\frac{1}{2}\pi}\frac{\cos x\log(\sin x)}{\sin x}\,dx = \tfrac{1}{2}\lim_{\varepsilon\to 0^+}\big(\log(\sin x)\big)^2\Big|_\varepsilon^{\frac{1}{2}\pi} = \infty,$$

since $\sin\varepsilon\to 0^+$, and therefore $\log(\sin\varepsilon)\to-\infty$, as $\varepsilon\to 0^+$.

e. Partial integration gives

$$\int_1^\infty \frac{\operatorname{arcsec} z}{z^3}\,dz = -\lim_{t\to\infty}\frac{\operatorname{arcsec} z}{2z^2}\Big|_1^t + \int_1^\infty\frac{dz}{2z^3\sqrt{z^2-1}} = \int_1^\infty\frac{dz}{2z^3\sqrt{z^2-1}},$$

since $0<(\operatorname{arcsec} t)/(2t^2) < \frac{1}{4}\pi t^{-2}$ if $t>1$, and $\frac{1}{4}\pi t^{-2}\to 0$ as $t\to\infty$. If $y=\sqrt{z^2-1}$ then $z^2=y^2+1$, $y\to 0^+$ as $z\to 1^+$, $y\to\infty$ as $z\to\infty$, $y\,dy = z\,dz$ and

$$d\Big(\frac{y}{z^2}\Big) = \frac{dy}{z^2} - \frac{2y}{z^3}\,dz = \frac{dy}{z^2} - \frac{2(z^2-1)}{z^3 y} = \frac{2}{z^3y}\,dz - \frac{dy}{y^2+1}.$$

Therefore, the integral in question is equal to

$$\int_1^\infty\frac{dz}{2z^3y} = \lim_{y\to\infty}\frac{y}{4(y^2+1)} + \frac{1}{4}\int_0^\infty\frac{dy}{y^2+1} = \frac{1}{4}\lim_{y\to\infty}\arctan(y) = \tfrac{1}{8}\pi.$$

f. If $x\geqslant 1$ and $\varphi = \operatorname{arccsc} x$, then $0<\varphi\leqslant\frac{1}{2}\pi$, so $0<x^{-1}=\sin\varphi<\varphi=\operatorname{arccsc} x$, and hence

$$\frac{\sqrt{\operatorname{arccsc} x}}{\sqrt[3]{x}} > \frac{\sqrt{x^{-1}}}{\sqrt[3]{x}} = \frac{1}{x^{5/6}} > 0. \qquad \text{But} \qquad \int_1^\infty\frac{dx}{x^{5/6}}$$

is a divergent improper integral ($p=\frac{5}{6}<1$ in the scale of powers at ∞), so the improper integral

$$\int_1^\infty\frac{\sqrt{\operatorname{arccsc} x}}{\sqrt[3]{x}}\,dx$$

is divergent, by the comparison principle.

g. If $x = w^{-1}\sqrt{w^2-2}$, then $x^2 = 1-2w^{-2}$ and $\frac{1}{2}x\,dx = w^{-3}\,dw$. If $w\to\sqrt{2}^+$ then $x\to 0^+$ and if $w\to\infty$ then $x\to 1^-$, so the impropriety is resolved by the change of variables. Since $w^{-2}=\frac{1}{2}(1-x^2)$, the integral in question is equal to

$$\int_{\sqrt{2}}^\infty\frac{w}{\sqrt{w^2-2}}\cdot\frac{1}{w^4}\cdot\frac{dw}{w^3} = \frac{1}{8}\int_0^1(x^2-1)^2\,dx = \frac{1}{8}\Big(\tfrac{1}{5}x^5 - \tfrac{2}{3}x^3 + x\Big)\Big|_0^1 = \frac{1}{8}\Big(\tfrac{1}{5}-\tfrac{2}{3}+1\Big) = \frac{1}{15}.$$

h. Since

$$\int_1^\infty\frac{dx}{x\sqrt{x^2-1}} = \lim_{\alpha\to\infty}\int_2^\alpha\frac{dx}{x\sqrt{x^2-1}} + \lim_{\beta\to 1^+}\int_\beta^2\frac{dx}{x\sqrt{x^2-1}} = \lim_{\alpha\to\infty}\operatorname{arcsec}\alpha - \lim_{\beta\to 1^+}\operatorname{arcsec}\beta = \tfrac{1}{2}\pi,$$

and if $t=\sqrt{x^2+1}$, then

$$\int_1^\infty\frac{dx}{x\sqrt{x^2+1}} = \lim_{\gamma\to\infty}\int_{\sqrt{2}}^\gamma\frac{dt}{t^2-1} = \tfrac{1}{2}\lim_{\gamma\to\infty}\log\frac{t-1}{t+1}\Big|_{\sqrt{2}}^\gamma = -\tfrac{1}{2}\log\frac{\sqrt{2}-1}{\sqrt{2}+1} = \log(1+\sqrt{2}),$$

it follows that

$$\int_1^\infty \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{x\sqrt{x^4-1}}\,dx = \int_1^\infty \frac{dx}{x\sqrt{x^2-1}} + \int_1^\infty \frac{dx}{x\sqrt{x^2+1}} = \tfrac{1}{2}\pi + \log(1+\sqrt{2}).$$

i. Partial integration (integrating the exponential factor) gives

$$\int e^{-x}\cos(x)\,dx = -e^{-x}\cos(x) + e^{-x}\sin(x) - \int e^{-x}\cos(x)\,dx = \tfrac{1}{2}e^{-x}(\sin(x)-\cos(x)) + C.$$

Therefore,

$$\int_0^\infty e^{-x}\cos(x)\,dx = \lim_{\alpha\to\infty}\int_0^\alpha e^{-x}\cos(x)\,dx = \tfrac{1}{2} + \tfrac{1}{2}\lim_{\alpha\to\infty}\frac{\sin(\alpha)-\cos(\alpha)}{e^\alpha} = \tfrac{1}{2},$$

since $0 \leqslant |e^{-\alpha}(\sin\alpha - \cos\alpha)| \leqslant \sqrt{2}\,e^{-\alpha}$, and $\sqrt{2}\,e^{-\alpha} \to 0$ as $\alpha \to \infty$.

j. The integral is a sum of limits of definite integrals as follows.

$$\int_0^\infty \frac{\arctan(x)}{x\sqrt{x}}\,dx = \lim_{\delta\to 0^+}\int_\delta^1 \frac{\arctan(x)}{x\sqrt{x}}\,dx + \lim_{\omega\to\infty}\int_1^\omega \frac{\arctan(x)}{x\sqrt{x}}\,dx.$$

Partial integration (integrating the power) gives

$$\int \frac{\arctan(x)}{x\sqrt{x}}\,dx = -\frac{2\arctan(x)}{\sqrt{x}} + 2\int \frac{dx}{(1+x^2)\sqrt{x}}$$

If $\omega > 0$ then

$$0 < \frac{2\arctan(\omega)}{\sqrt{\omega}} < \frac{\pi}{\sqrt{\omega}}, \qquad \text{and thus} \qquad \lim_{\omega\to\infty}\frac{2\arctan\omega}{\sqrt{\omega}} = 0 \qquad \text{because} \qquad \lim_{\omega\to\infty}\frac{\pi}{\sqrt{\omega}} = 0.$$

Also, if $\delta > 0$ then $0 < \arctan\delta < \frac{1}{2}\pi$, so $0 < \arctan\delta < \tan(\arctan\delta) = \delta$, and hence

$$0 < \frac{2\arctan\delta}{\sqrt{\delta}} < \frac{2\delta}{\sqrt{\delta}} = 2\sqrt{\delta}, \qquad \text{so} \qquad \lim_{\delta\to 0^+}\frac{-2\arctan\delta}{\sqrt{\delta}} = 0, \qquad \text{because} \qquad \lim_{\delta\to 0^+}(2\sqrt{\delta}) = 0.$$

Therefore,

$$\int_0^\infty \frac{\arctan x}{x\sqrt{x}}\,dx = 2\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}}.$$

If $y = \sqrt{(2x)}$, then $x = \frac{1}{2}y^2$ and $dx = y\,dy$. Also $y \to 0$ as $x \to 0^+$ and $y \to \infty$ as $x \to \infty$, so

$$2\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}} = 2\sqrt{2}\int_0^\infty \frac{y\,dy}{\left(1+\frac{1}{4}y^4\right)y} = 8\sqrt{2}\int_0^\infty \frac{dy}{y^4+4},$$

where now the discontinuity at the origin is removable. The denominator of the integrand factorizes as $y^4+4 = (y^2+2y+2)(y^2-2y+2)$, and in the resolution into partial fractions,

$$\frac{ay+b}{y^2+2y+2} + \frac{cy+d}{y^2-2y+2} = \frac{8}{y^4+4},$$

clearing denominators gives $(ay+b)(y^2-2y+2)+(cy+d)(y^2+2y+2) = 8$. Comparing coefficients gives

$$a+c=0, \qquad -2a+b+2c+d=0, \qquad 2a-2b+2c+2d=0 \qquad \text{and} \qquad 2b+2d=8.$$

Subtracting the first equation from half the third equation gives $-b+d=0$, and one-half of the fourth equation is $b+d=4$, so $b=d=2$. The second equation now gives $-a+c=-2$, so $a=1$ and $c=-1$ using the first equation. Now

$$\int \frac{y+2}{y^2+2y+2}\,dy = \int \frac{y+1}{y^2+2y+2}\,dy + \int \frac{dy}{(y+1)^2+1} = \tfrac{1}{2}\log(y^2+2y+2) + \arctan(y+1) + C,$$

and

$$\int \frac{y-2}{y^2-2y+2}\,dy = \int \frac{y-1}{y^2+2y+2}\,dy - \int \frac{dy}{(y-1)^2+1} = \tfrac{1}{2}\log(y^2-2y+2) - \arctan(y-1) + C.$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{\arctan x}{x\sqrt{x}}\,dx &= 8\sqrt{2}\int_0^\infty \frac{dy}{y^4+4}\\ &= \sqrt{2}\lim_{\alpha\to\infty}\left\{\tfrac{1}{2}\log\frac{y^2+2y+2}{y^2-2y+2} + \arctan(y+1) + \arctan(y-1)\right\}\Bigg|_0^\alpha\\ &= \sqrt{2}\left\{\left(0+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi\right) - \left(0+\tfrac{1}{4}\pi-\tfrac{1}{4}\pi\right)\right\}\\ &= \pi\sqrt{2}.\end{aligned}$$

k. Resolving the integrand into partial fractions, integrating, and collecting the logarithms, gives

$$\begin{aligned}\int_2^\infty \frac{4x+3}{(2x-1)(x+2)(x-1)}\,dx &= \int_2^\infty\left\{\frac{-4}{2x-1} - \frac{1/3}{x+2} + \frac{7/3}{x-1}\right\}dx\\ &= \tfrac{1}{3}\lim_{\beta\to\infty}\log\frac{(x-1)^7}{(2x-1)^6(x+2)}\Bigg|_2^\beta\\ &= \tfrac{1}{3}\left\{\log\frac{1}{2^6} - \log\frac{1}{3^6\cdot 4}\right\} = \tfrac{1}{3}\log\frac{9^3}{2^4}\\ &= \log\left(\tfrac{9}{4}\sqrt[3]{4}\right).\end{aligned}$$

l. The denominator of the integrand is $x^3-3x+2 = (x-1)^2(x+2)$, so the integrand has infinite discontinuities at -2 and 1, each of which falls in the interval of integration (so the improper integral is the sum of four limits of definite integrals). If $1 < x \leqslant 2$, then $3 < 2x+1 \leqslant 5$ and $3 < x+2 \leqslant 4$, so

$$\frac{2x+1}{\sqrt[3]{(x^3-3x+2)^2}} = \frac{2x+1}{(x+2)^{2/3}(x-1)^{4/3}} > \frac{3}{4^{2/3}}\cdot\frac{1}{(x-1)^{4/3}}, \qquad \text{and} \qquad \int_1^2 \frac{dx}{(x-1)^{4/3}} = \int_0^1 \frac{dx}{x^{4/3}}$$

is a divergent improper integral ($p = \frac{4}{3} > 1$ in the scale of powers at 0), so the comparison principle implies that improper integral

$$\int_1^2 \frac{2x+1}{\sqrt[3]{(x^3-3x+2)^2}}\,dx, \qquad \text{and therefore also} \qquad \int_{-3}^2 \frac{2x+1}{\sqrt[3]{(x^3-3x+2)^2}}\,dx,$$

is divergent. (The indefinite integral could have been evaluated using a rationalizing change of variables, but in this case that is not necessary.)

Solution to Exercise 3. — First notice that if $r \leqslant 0$ and k is a positive integer then $|x^{-r}\sin x| \geqslant |\sin x|$ on $[k\pi(k+1)\pi]$, and so direct evaluation of the corresponding definite integral of the sine function gives

$$\int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x^r}\,dx \quad \begin{cases} \geqslant 2 & \text{if } k \text{ is even, and} \\ \leqslant -2 & \text{if } k \text{ is odd.} \end{cases}$$

So if $r \leqslant 0$, then the integral in question (oscillates and) diverges. On the other hand, recall that

$$\cos x < \frac{\sin x}{x} < 1, \qquad \text{if } 0 < x < \tfrac{1}{2}\pi,$$

and so

$$\tfrac{1}{2} < \frac{\sin x}{x}, \qquad \text{and therefore} \qquad \frac{1}{2x^{r-1}} < \frac{\sin x}{x^r}, \qquad \text{if} \quad 0 < x < \tfrac{1}{3}\pi.$$

It follows that

$$\int_0^1 \frac{\sin x}{x^r}\,dx \qquad \text{diverges with} \qquad \tfrac{1}{2}\int_0^1 \frac{dx}{x^{r-1}}$$

by the comparison principle (using the scale of powers near the origin) if $r \geqslant 2$ (since then $r-1 \geqslant 1$). So the integral in question diverges if $r \leqslant 0$ or $r \geqslant 2$. Now write

$$\int_0^\infty \frac{\sin x}{x^r}\,dx = \int_0^{\frac{1}{2}\pi} \frac{\sin x}{x^r}\,dx + \int_{\frac{1}{2}\pi}^\infty \frac{\sin x}{x^r}\,dx.$$

Since

$$\frac{\sin x}{x} < 1, \qquad \text{and therefore} \qquad \frac{\sin x}{x^r} < \frac{1}{x^{r-1}}, \qquad \text{if } 0 < x < \tfrac{1}{2}\pi,$$

it follows that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{x^r}\,dx \qquad \text{converges with} \qquad \int_0^{\frac{1}{2}\pi} \frac{dx}{x^{r-1}}$$

by the comparison principle (using the scale of powers near the origin) if $r < 2$ (since then $r-1 < 1$). Next, observe that if $r > 0$ then

$$\int_{\frac{1}{2}\pi}^\infty \frac{\sin x}{x^r}\,dx = \lim_{t\to\infty}\left\{-\frac{\cos x}{x^r}\right\}\Bigg|_{\frac{1}{2}\pi}^{t} - r\int_{\frac{1}{2}\pi}^\infty \frac{\cos x}{x^{r+1}}\,dx = -r\int_{\frac{1}{2}\pi}^\infty \frac{\cos x}{x^{r+1}}\,dx$$

by partial integration (as $|x^{-r}\cos x| \leqslant x^{-r}$ if $x > 0$, and $x^{-r} \to 0$ as $x \to \infty$). Now $|x^{-(r+1)}\cos x| \leqslant x^{-(r+1)}$ if $x > 0$, and so

$$\int_{\frac{1}{2}\pi}^\infty \frac{\cos x}{x^{r+1}}\,dx \qquad \text{converges with} \qquad \int_{\frac{1}{2}\pi}^\infty \frac{dx}{x^{r+1}}$$

by the comparison principle (using the scale of powers at ∞) if $r > 0$ (since then $r+1 > 1$). Therefore

$$\int_{\frac{1}{2}\pi}^\infty \frac{\sin x}{x^r}\,dx$$

converges if $r > 0$. It follows that

$$\int_0^\infty \frac{\sin x}{x^r}\,dx$$

is convergent if, and only if, $0 < r < 2$.

Solution to Exercise 4. — The curves meet where

$$0 = x^2 - \frac{2x}{x^2+1} = \frac{x^4+x^2-2x}{x^2+1} = \frac{x(x-1)(x^2+x+2)}{x^2+1}.$$

Since $x^2+x+2 = \frac{1}{4}\big((2x+1)^2+7\big)$, has no real zeros, the curves meet at the points $(0,0)$ and $(1,1)$. If $0 < x < 1$ then, by the displayed factorization,

$$x^2 < \frac{2x}{x^2+1}.$$

Therefore, the area of the region enclosed by the given curves is equal to

$$\int_0^1 \left\{\frac{2x}{x^2+1} - x^2\right\}dx = \left\{\log(x^2+1) - \tfrac{1}{3}x^3\right\}\Big|_0^1 = \log 2 - \tfrac{1}{3}.$$

Solution to Exercise 5. — Let

$$y = (x+1)(x^2-5x+6)(x^2+3x-10) \qquad \text{and} \qquad y' = 5(x-2)^2(x^2-1);$$

then

$$\begin{aligned} y - y' &= (x+1)(x^2-5x+6)(x^2+3x-10) - 5(x-2)^2(x^2-1) \\ &= (x+5)(x+1)(x-2)^2(x-3) - 5(x+1)(x-1)(x-2)^2 \\ &= (x+1)(x-2)^2(x^2-3x-10) \\ &= (x+2)(x+1)(x-2)^2(x-5). \end{aligned}$$

Therefore, the curves meet where x is -2, -1, 2 and 5. Analyzing the sign of the factors of $y - y'$ reveals that

$$y > y' \quad \text{if} \quad -2 < x < -1, \qquad \text{and} \qquad y \leqslant y' \quad \text{if} \quad -1 < x < 5.$$

Therefore, the area of the region enclosed by the curves is equal to

$$\int_{-2}^{-1}(y-y')\,dx + \int_{-1}^{5}(y'-y)\,dx,$$

or

$$\int_{-2}^{-1}(x+2)(x+1)(x-2)^2(x-5)\,dx + \int_{-1}^{5}(x+2)(x+1)(x-2)^2(5-x)\,dx.$$

Solution to Exercise 6. — The curves meet where

$$x^2 + x = 3x - x^3, \qquad \text{or} \qquad x(x+2)(x-1) = 0.$$

If $-2 < x < 0$, the quadratic function is larger, and if $0 < x < 1$, the cubic function is larger.

a. The area of $\mathscr{R}$ is equal to

$$\begin{aligned} \int_{-2}^{0}(x^3+x^2-2x)\,dx + \int_{1}^{0}(x^3+x^2-2x)\,dx &= \left\{\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - x^2\right\}\Big|_{-2}^{0} + \left\{\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 - x^2\right\}\Big|_{1}^{0} \\ &= \tfrac{8}{3} + \tfrac{5}{12} = \tfrac{37}{12}. \end{aligned}$$

b. If $\mathscr{R}_2$ is revolved about the line defined by $y = 2$, then the cross sections perpendicular to the x-axis of the generated solid are annuli of inner radius $2 - 3x + x^3$ and outer radius $2 - x - x^2$, for $0 \leqslant x \leqslant 1$. So the volume of the solid is equal to

$$\pi \int_0^1 \left\{ (2 - x - x^2)^2 - (2 - 3x + x^3)^2 \right\} dx.$$

c. If $\mathscr{R}_1$ is revolved about the line defined by $x = 1$ then the resulting solid consists of concentric cylindrical shells of radius $1 - x$ and height $x^3 + x^2 - 2x$, for $-2 \leqslant x \leqslant 0$. So the volume of the solid is equal to

$$2\pi \int_{-2}^0 (1-x)(x^3 + x^2 - 2x)\, dx = 2\pi \int_{-2}^0 (-x^4 + 3x^2 - 2x)\, dx = 2\pi \left\{ -\tfrac{1}{5}x^5 + x^3 - x^2 \right\} \Big|_{-2}^0 = \tfrac{56}{5}\pi.$$

d. The length of the perimeter of $\mathscr{R}$ is equal to the sum of the lengths of the curves on $[-2, 1]$, or

$$\int_{-2}^1 \left\{ \sqrt{1 + (2x+1)^2} + \sqrt{1 + 9(x^2-1)^2} \right\} dx.$$

Solution to Exercise 7. — The volume of the solid obtained by revolving $\mathscr{R}$ about the line defined by $y = 5$ is

$$39\pi = \pi \int_2^7 \left\{ \left(5 - f(x)\right)^2 - \left(5 - g(x)\right)^2 \right\} dx = \pi \int_2^7 \left\{ \left(g(x) - f(x)\right)\left(10 - f(x) - g(x)\right) \right\} dx.$$

The volume of the solid obtained by revolving $\mathscr{R}$ about the line defined by $y = -1$ is

$$69\pi = \pi \int_2^7 \left\{ \left(g(x) + 1\right)^2 - \left(f(x) + 1\right)^2 \right\} dx = \pi \int_2^7 \left\{ \left(g(x) - f(x)\right)\left(f(x) + g(x) + 2\right) \right\} dx.$$

The sum of these volumes is

$$108\pi = 12\pi \int_2^7 \left(g(x) - f(x)\right) dx, \qquad \text{and so} \qquad \int_2^7 \left(g(x) - f(x)\right) dx = 9$$

is the area of $\mathscr{R}$.

Solution to Exercise 8. — a. If $x = \ln(1 - y^2)$, then

$$\frac{dx}{dy} = \frac{2y}{y^2 - 1}, \qquad \text{and so} \qquad \left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{4y^2}{(1-y^2)^2} = \frac{(1-y^2)^2 + 4y^2}{(1-y^2)^2} = \frac{(1+y^2)^2}{(1-y^2)^2}.$$

Therefore, the length of the curve in question is equal to

$$\int_0^{\frac{1}{2}} \frac{1+y^2}{1-y^2}\, dy = \int_0^{\frac{1}{2}} \left\{ -1 + \frac{2}{1-y^2} \right\} dy = \left\{ -y - \log \frac{1-y}{1+y} \right\} \Big|_0^{\frac{1}{2}} = -\tfrac{1}{2} - \log \tfrac{1}{3} = \log 3 - \tfrac{1}{2}.$$

b. If $y = \log x$, then

$$\frac{dy}{dx} = \frac{1}{x}, \qquad \text{and so} \qquad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 1/x^2} = \frac{\sqrt{x^2+1}}{x},$$

since x is positive on $[1, \sqrt{2}]$. Therefore, the length of the curve is

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{\sqrt{x^2+1}}{x}\, dx = \int_{\sqrt{2}}^{\sqrt{3}} \frac{t^2}{t^2-1}\, dt = \int_{\sqrt{2}}^{\sqrt{3}} \left\{ 1 + \frac{1}{t^2-1} \right\} dt &= \left\{ t + \tfrac{1}{2} \log \frac{t-1}{t+1} \right\} \Big|_{\sqrt{2}}^{\sqrt{3}} \\ &= \sqrt{3} - \sqrt{2} + \tfrac{1}{2} \log\left(\frac{\sqrt{3}-1}{\sqrt{3}+1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}-1} \right) \\ &= \sqrt{3} - \sqrt{2} + \log\left((\sqrt{3}-1)(\sqrt{2}+1) \right), \end{aligned}$$

where $t = \sqrt{x^2+1}$.

c. If $y = \log(\cos x)$, then

$$\frac{dy}{dx} = -\tan(x), \qquad \text{and so} \qquad \frac{ds}{dx} = \sqrt{1 + \tan^2(x)} = \sec x,$$

and the length of the curve is

$$\int_0^{\frac{1}{3}\pi} \sec(x)\, dx = \log(\sec x + \tan x) \Big|_0^{\frac{1}{3}\pi} = \log(2 + \sqrt{3}).$$

Solution to Exercise 9. — The area of $\mathscr{R}$ is equal to

$$\int_0^\pi \sqrt{1 - \cos(x)}\, dx = \sqrt{2} \int_0^\pi \sin\left(\tfrac{1}{2}x\right) dx = -2\sqrt{2} \cos\left(\tfrac{1}{2}x\right) \Big|_0^\pi = 2\sqrt{2}.$$

If $\mathscr{R}$ is revolved around the x-axis then cross sections perpendicular to the x-axis of the resulting solid are circular disks of of radius $\sqrt{1 - \cos(x)}$, for $0 \leqslant x \leqslant \pi$. So the volume of the solid is equal to

$$\pi \int_0^\pi (1 - \cos(x))\, dx = \pi^2.$$

If $\mathscr{R}$ is revolved about the y-axis then the resulting solid consists of concentric cylindrical shells of radius x and height $\sqrt{1 - \cos(x)}$, for $0 \leqslant x \leqslant \pi$. So the volume of the solid is equal to

$$\begin{aligned} 2\pi \int_0^\pi x\sqrt{1 - \cos(x)}\, dx = 2\pi\sqrt{2} \int_0^\pi x \sin\left(\tfrac{1}{2}x\right) dx &= 2\pi\sqrt{2} \left\{ -2x \cos\left(\tfrac{1}{2}x\right) \right\} \Big|_0^\pi + 4\pi\sqrt{2} \int_0^\pi \cos\left(\tfrac{1}{2}x\right) dx \\ &= 8\pi\sqrt{2} \sin\left(\tfrac{1}{2}x\right) \Big|_0^\pi = 8\pi\sqrt{2}. \end{aligned}$$

If $\mathscr{R}$ is revolved about the line defined by $y = 2$, then cross sections perpendicular to the x-axis of the generated solid are annuli of inner radius $2 - \sqrt{1 - \cos(x)}$ and out radius 2, for $0 \leqslant x \leqslant \pi$. So the volume of the solid is equal to

$$\pi \int_0^\pi \left\{ 4 - \left(2 - \sqrt{1 - \cos(x)}\right)^2 \right\} dx = 4\pi \int_0^\pi \sqrt{1 - \cos(x)}\, dx - \pi \int_0^\pi (1 - \cos(x))\, dx = 8\pi\sqrt{2} - \pi^2.$$

Solution to Exercise 10. — A four dimensional sphere of radius r consists of sets

$$\{(x,y,z)\colon x^2+y^2+z^2 \leqslant r^2-t^2\}, \qquad \text{for} \qquad -r \leqslant t \leqslant r,$$

each of which is a three dimensional sphere of radius $\sqrt{r^2-t^2}$ and hence volume $\frac{4}{3}\pi(r^2-t^2)^{3/2}$. Thus, the volume of a four dimensional sphere of radius r is equal to

$$\frac{4}{3}\pi\int_{-r}^{r}(r^2-t^2)^{3/2}\,dt = \frac{8}{3}\pi r^4\int_0^1(1-p^2)^{3/2}\,dp,$$

by symmetry and the change of variables $t=pr$. Partial integration gives

$$\int_0^1(1-p^2)^{3/2}\,dp = p(1-p^2)^{3/2}\Big|_0^1 + 3\int_0^1 p^2(1-p^2)^{1/2}\,dp = 3\int_0^1(p^2-1+1)(1-p^2)^{1/2}\,dp$$

$$= -3\int_0^1(1-p^2)^{3/2}\,dp + 3\int_0^1(1-p^2)^{1/2}\,dp = \frac{3}{4}\int_0^1(1-p^2)^{1/2}\,dp,$$

where the first term on the right in the second line is absorbed by the left hand side. Since

$$\int_0^1(1-p^2)^{1/2}\,dp = \frac{1}{4}\pi$$

is one quarter the area of a unit circle, it follows that the volume of a four dimensional sphere of radius r is equal to $\frac{8}{3}\pi r^4\cdot\frac{3}{4}\cdot\frac{1}{4}\pi = \frac{1}{2}\pi^2 r^4$.
