An old Test 3

Question 1. — Give the explicit solution of the initial value problem

$$e^t(1-t)^2\frac{dx}{dt} = t(1-x)^2; \quad x(0)=0.$$

Question 2. — One model for the spread of an epidemic is that the rate of spread is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of five thousand inhabitants, one hundred and sixty people have a disease at the beginning of the week and twelve hundred have it at the end of the week. How long does it take for eighty per cent of the population to become infected?

Question 3. — Evaluate each of the following limits, and simplify the results.

a. $\lim\limits_{t\to 0}\left\{\frac{1}{t}-\frac{1}{e^t-1}\right\}$ b. $\lim\limits_{\mu\to 0}\left(\frac{\sin\mu}{\mu}\right)^{1/\mu^2}$ c. $\lim\limits_{x\to\infty}\left\{e^{-2x}\left(1+\frac{2}{x}\right)^{x^2}\right\}$

Question 4. — Find the limit of each sequence or explain why it diverges.

a. $\left\{(-1)^k k\operatorname{arccsc}(k)\right\}$ b. $\left\{\log(n^2+1)-2\log(3n-2)\right\}$ c. $\left\{\frac{(n!)^2\sin(n^n)}{(2n)!}\right\}$

Question 5. — The sequence $\{a_k\}$ is defined by $a_0=1$ and $a_{k+1}=1-\frac{1}{2+a_k}$ for $k\geqslant 0$.

a. Prove that $0<a_{k+1}<a_k\leqslant 1$ for all integers $k\geqslant 0$.
b. Explain why the sequence $\{a_k\}$ is convergent, and find its limit.

Question 6. — Find the sum of the series or explain why it is divergent.

a. $\sum\limits_{k=1}^{\infty}\frac{4^{2k-1}-2\cdot 5^k}{15^k}$ b. $\sum\limits_{n=0}^{\infty}\left\{\arctan(\sqrt{2n+1})-\arctan(\sqrt{2n+5})\right\}$ c. $\sum\limits_{j=3}^{\infty}\frac{13j-6}{j(j-2)(j+3)}$

Question 7. — For each series, determine whether it is convergent or divergent. To earn full credit write complete and proper solutions, and justify all assertions carefully.

a. $\sum\limits_{k=0}^{\infty}\frac{\sqrt{5k^3-2k+7}}{k+\sqrt[3]{3k^5-k+7}}$ b. $\sum\limits_{n=1}^{\infty}\frac{n^{3n}}{(3n)!}$ c. $\sum\limits_{\mu=1}^{\infty}\left(1-\frac{1}{\sqrt{\mu}}\right)^{\mu}$ d. $\sum\limits_{n=2}^{\infty}\frac{\csc^2(n)}{\log(n)}$

Question 8. — Determine whether each series is absolutely convergent, conditionally convergent or divergent. To earn full credit display complete and proper solutions, and justify all assertions carefully.

a. $\sum\limits_{k=1}^{\infty}\frac{\sin(k)(\log k)^3}{k^2-k+1}$ b. $\sum\limits_{n=0}^{\infty}\frac{(-1)^n n^3}{e^{\sqrt[3]{n}}}$ c. $\sum\limits_{j=1}^{\infty}\left(\frac{2-3j}{2j+5}\right)^{j/3}$ d. $\sum\limits_{k=1}^{\infty}(-1)^k\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}$

Question 9. — Find the radius and interval of convergence of the power series

$$\sum_{k=2}^{\infty}\frac{\tan(1/k)}{2^k(\log k)^{3/2}}(3x-5)^k.$$

Question 10. — a. Find the Taylor series of $\sqrt{3x+1}$ centred at 1. Express the general term of the series without using ellipses (*i.e.*, without "$\cdots$"), write the first four terms of the series explicitly, and give the interval of convergence of the series.

b. Find the Maclaurin series of

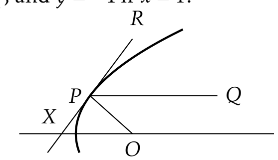
$$\int_0^x f(t)\,dt, \qquad \text{where} \qquad f(x)=\begin{cases}\frac{\sin^2(2x)}{x} & \text{if } x\neq 0, \text{ and}\\ 0 & \text{if } x=0.\end{cases}$$

c. The power series $g(x)=\sum_{k=0}^{\infty}c_k(x+2)^k$ converges if $x=3$ and diverges if $x=-8$. What can you conclude about the convergence of: i. $g(0)$? ii. $g(7)$? iii. $g(-7)$? Explain your answers.

Another old Test 3

Question 1. — Express y as a function of x given that $\frac{dy}{dx}=\frac{x+1}{xy+x}$, and $y=-4$ if $x=1$.

Question 2. — Find all curves with the property that if a line segment is drawn from a fixed point O to any point P on the curve, then the acute angle between the tangent at P and the segment is equal to the acute angle between the tangent at P and the horizontal line through P. (In the figure, $\angle XPO=\angle QPR$.)



Question 3. — Evaluate each of the following limits, and simplify the results.

a. $\lim\limits_{x\to\infty}\left\{x^2\log\left(\cos(1/x)\right)\right\}$ b. $\lim\limits_{x\to 0}\frac{\sqrt[x]{1+x}-e}{x}$ c. $\lim\limits_{x\to\frac{1}{2}\pi}(\sin 5x)^{\tan^2 x}$

Question 4. — a. Let $a_n=\log(n+1)-\log(2n+1)$, and let $\{b_n\}_{n\geqslant 1}$ satisfy $b_1+\cdots+b_n=a_n$, for $n\geqslant 1$. Determine the value or limit, or explain the divergence, of $\{a_n\}$, $\{b_n\}$, $\sum\limits_{n=1}^{\infty}a_n$ and $\sum\limits_{n=1}^{\infty}b_n$.

b. Find the limit or explain the divergence of: i. $\left\{\frac{k^k}{k!}\right\}$; ii. $\left\{(-1)^n\csc(\pi^{-n})\left(\frac{1}{2}\pi-\operatorname{arcsec}(\pi^{n+1})\right)\right\}$.

Question 5. — Find the sum of each series or else explain why it is divergent.

a. $\sum\limits_{k=2}^{\infty}\log\left(\frac{8k^2-2k-15}{8k^2-2k-3}\right)$ b. $\frac{3}{8}+\frac{1}{1\cdot 4}-\frac{9}{32}+\frac{1}{2\cdot 5}+\frac{27}{128}+\frac{1}{3\cdot 6}-\frac{81}{512}+\frac{1}{4\cdot 7}+\cdots$

Question 6. — For each series, determine whether it is convergent or divergent. State which test you are using and verify that the conditions for using it are satisfied.

a. $\sum\limits_{k=1}^{\infty}2^{2k}\left(\frac{k-3}{k}\right)^{k^2}$ b. $\sum\limits_{m=0}^{\infty}\frac{\pi+e\sin(m)}{\sqrt{m^2+1}}$ c. $\sum\limits_{n=0}^{\infty}\frac{2^{n-2}+5^{n+1}}{3^n+7^{n-1}}$ d. $\sum\limits_{n=0}^{\infty}\frac{(4n)!}{2^{4n}((2n)!)^2}$

Question 7. — Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify all assertions carefully.

a. $\sum\limits_{n=1}^{\infty}\frac{\cos(\pi n)\operatorname{arccot}(n)}{\sqrt[3]{n}}$ b. $\sum\limits_{k=2}^{\infty}(-1)^k\frac{\sin(1/k)}{\log(k)}$ c. $\sum\limits_{k=1}^{\infty}\frac{(-3)^{3k}(k!)^3}{(3k)!}$ d. $\sum\limits_{k=3}^{\infty}\frac{(-1)^k}{n^{1+1/(\log(\log k))}}$

Question 8. — Find the radius and interval of convergence of the power series

$$\sum_{k=1}^{\infty}\frac{(-1)^k\log(k)}{2^k\sqrt[5]{k}}(3x-2)^k.$$

Question 9. — Let $\{a_n\}$ be a sequence of positive real numbers such that

$$\frac{1}{2n}<a_n<\frac{1}{n}, \qquad \text{if} \quad n\geqslant 1. \qquad (*)$$

a. Show that if $a_{n+1}\leqslant a_n$ for $n\geqslant 1$, then the series $\sum\limits_{n=1}^{\infty}(-1)^n a^n$ is conditionally convergent.

b. Give an example of a sequence $\{a_n\}$ which satisfies $(*)$, yet the series $\sum\limits_{n=1}^{\infty}(-1)^n a^n$ is divergent.

Question 10. — a. Compute the Taylor series of $\sin\left(\frac{1}{3}x\right)\cos\left(\frac{1}{3}x\right)$ centred at $\frac{3}{4}\pi$.

b. Compute the Taylor series of $(x+2)\log(8-4x+x^2)$ centered at 2.
c. Use the Maclaurin series of arctan to compute $f^{(2018)}(0)$, where $f(x)=5x^6\arctan(3x^4)$.

Solutions to an old Test 3

Solution to Question 1. — The differential equation is equivalent to

$$\frac{1}{(1-x)^2}\frac{dx}{dt}=\frac{te^{-t}}{(1-t)^2},$$

and integrating (using partial integration on the right side) gives

$$\frac{1}{1-x}=\frac{te^{-t}}{1-t}-\int\frac{e^{-t}(1-t)}{1-t}\,dt=\frac{te^{-t}}{1-t}+e^{-t}+C=\frac{e^{-t}}{1-t}+C.$$

If $x=0$ when $t=0$, $1=1+C$, or $C=0$. Therefore, $1-x=e^t(1-t)$, or $x=1-e^t(1-t)$.

Solution to Question 2. — Let y denote the number of people infected after t weeks. Then using the given model there is a real number α such that

$$\frac{dy}{dt}=\alpha y(1-y/p),\qquad \text{where}\quad p=5000$$

is the population of the town. Separating variables and integrating gives

$$\frac{p/y^2}{p/y-1}\frac{dy}{dt}=\alpha,\qquad i.e.,\qquad \log\left|\frac{p}{y}-1\right|=c-\alpha t,\qquad \text{or}\qquad \frac{p}{y}-1=Ae^{-\alpha t}$$

(where $A=e^c$, since $0<y<p$). Initially,

$$A=\tfrac{5000}{160}-1=\tfrac{121}{4},\quad \text{and after one week}\quad \tfrac{121}{4}e^{-\alpha}=\tfrac{5000}{1200}-1=\tfrac{19}{6},\quad \text{so}\quad e^{-\alpha}=\tfrac{38}{363}.$$

Therefore, the eighty per cent of the inhabitants have become infected when

$$\tfrac{121}{4}\left(\tfrac{38}{363}\right)^t=\tfrac{10}{8}-1=\tfrac{1}{4},\qquad \text{or after approximately}\qquad t=\frac{\log 121}{\log(38/363)}\ \text{weeks}.$$

Solution to Question 3. — a. Since $e^t=1+t+\frac12 t^2+\frac16 t^3+\&c.$, combining terms gives

$$\lim_{t\to0}\left\{\frac1t-\frac{1}{e^t-1}\right\}=\lim_{t\to0}\frac{e^t-t-1}{t(e^t-1)}=\lim_{t\to0}\frac{\frac12t^2+\frac16t^3+\&c.}{t^2+\frac12t^3+\&c.}=\frac12.$$

b. First notice that

$$\lim_{\mu\to0}\left(\frac{\sin\mu}{\mu}\right)^{\frac{1}{\frac{\sin\mu}{\mu}-1}}=\lim_{\mu\to0}\left(1+\frac{\sin\mu}{\mu}-1\right)^{\frac{1}{\frac{\sin\mu}{\mu}-1}}=\lim_{t\to0}(1+t)^{1/t}=e,$$

where $t=\frac{\sin\mu}{\mu}-1$. Next, since $\sin\mu=\mu-\frac16\mu^3+\frac{1}{120}\mu^5-\&c.$, it follows that

$$\lim_{\mu\to0}\frac{\frac{\sin\mu}{\mu}-1}{\mu^2}=\lim_{\mu\to0}\frac{\sin\mu-\mu}{\mu^2}=\lim_{\mu\to0}\frac{-\frac16\mu^3+\frac{1}{120}\mu^5+\&c.}{\mu^3}=-\frac16,$$

and therefore

$$\lim_{\mu\to0}\left(\frac{\sin\mu}{\mu}\right)^{1/\mu^2}=e^{-1/6}.$$

c. If $t=2/x$ and $x>2$ then $\log(1+2/x)=\log(1+t)=t-\frac12t^2+\frac13t^3-\frac14t^4+\&c.$, and the expression in the limit is the exponential function applied to

$$-2x+x^2\log(1+2/x)=-\frac4t+\frac{4}{t^2}\left(t-\tfrac12t^2+\tfrac13t^3-\tfrac14t^4+\&c.\right)=-2+\tfrac43t-t^2+\&c.,$$

which tends to -2 as $x\to\infty$ (since $t\to0$). Therefore, $\lim_{x\to\infty}\left\{e^{-2x}\left(1+\dfrac2x\right)^{x^2}\right\}=e^{-2}$.

Solution to Question 4. — a. If $t=\operatorname{arccsc}(k)$ and $k\geqslant1$, then then $k=(\sin t)^{-1}$, and so

$$\lim\{k\operatorname{arccsc}(k)\}=\lim_{t\to0^+}\frac{t}{\sin t}=1.$$

Since $\lim\{k\operatorname{arccsc}(k)\}$ is a non-zero real number, the sequence $\{(-1)^k k\operatorname{arccsc}(k)\}$ is divergent.

b. Combining the logarithms and revising gives

$$\lim\left\{\log(n^2+1)-2\log(3n-2)\right\}=\lim\log\left\{\frac{n^2+1}{(3n-2)^2}\right\}=\lim\log\left\{\frac{1+n^{-2}}{(3-2n^{-1})^2}\right\}=\log\left(\tfrac19\right).$$

c. If $0<\varepsilon<2$ and $n>-\log(\varepsilon)/\log(2)=-\log_2(\varepsilon)$, then

$$\left|\frac{(n!)^2\sin(n^n)}{(2n)!}\right|<\frac{1}{2n+1}\cdot\frac{2}{2n+2}\cdot\frac{3}{2n+3}\cdots\frac{n}{2n}<2^{-n}<\varepsilon,$$

so by definition the sequence in question converges to zero.

Solution to Question 5. — a. Since $a_0=1$ and $a_1=1-\frac13=\frac23$, it is clear that $0<a_1<a_0\leqslant1$. Next, if $k\geqslant0$ is an integer and $0<a_{k+1}<a_k\leqslant1$, then $2<2+a_{k+1}<2+a_k\leqslant3$, so

$$\tfrac13\leqslant\frac{1}{2+a_k}<\frac{1}{2+a_{k+1}}<\tfrac12,\qquad\text{and hence}\qquad \tfrac12<1-\frac{1}{2+a_{k+1}}<1-\frac{1}{2+a_k}\leqslant\tfrac23.$$

In other words, $\frac12<a_{k+2}<a_{k+1}\leqslant\frac23$, and so $0<a_{k+2}<a_{k+1}\leqslant1$. By the principle of mathematical induction, it follows that $0<a_{k+1}<a_k\leqslant1$ for every integer $k\geqslant0$.

b. From the solution to Part a, it follows that the sequence $\{a_k\}$ is bounded and decreasing, so the monotonic sequence theorem implies that $\{a_k\}$ converges to some real number α. Moreover, elementary properties of limits imply that $0\leqslant\alpha<1$ and

$$\alpha=\lim a_{k+1}=\lim\left\{1-\frac{1}{2+a_k}\right\}=1-\frac{1}{2+\alpha}.$$

This last equation is equivalent to $\alpha^2+\alpha=1$, or $(2\alpha+1)^2=5$, so $\alpha=-\frac12+\frac12\sqrt5$.

Solution to Question 6. — a. Since

$$\sum_{k=1}^{\infty}\tfrac14\left(\tfrac{16}{15}\right)^k\quad\text{is divergent,}\quad\text{and}\quad\sum_{k=1}^{\infty}2\left(\tfrac13\right)^k\quad\text{is convergent,}$$

(each series is a geometric series; the ratio of the first series is $\frac{16}{15}$, which is larger than 1, and the ratio of the second series is $\frac13$, which is positive and smaller than 1), it follows that the series

$$\sum_{k=1}^{\infty}\frac{4^{2k-1}-2\cdot5^k}{15^k}=\sum_{k=1}^{\infty}\left\{\tfrac14\left(\tfrac{16}{15}\right)^k-2\left(\tfrac13\right)^k\right\}$$

is divergent.

b. If

$$b_n=\arctan(\sqrt{2n+1})-\arctan(\sqrt{2n+5}),$$

and

$$B_n=\arctan(\sqrt{2n+1})+\arctan(\sqrt{2n+3}),$$

then $b_n=B_n-B_{n+1}$, for $n\geqslant0$, so

$$b_0+b_1+\cdots+b_{n-1}=(B_0-B_1)+(B_1-B_2)+\cdots+(B_{n-1}-B_n)=B_0-B_n.$$

Since $B_0=\arctan(1)+\arctan(\sqrt3)=\frac14\pi+\frac13\pi=\frac{7}{12}\pi$, and $\lim B_n=\pi$, the sum of the series is

$$\sum_{n=0}^{\infty}b_n=\lim(B_0-B_n)=\tfrac{7}{12}\pi-\pi=-\tfrac{5}{12}\pi.$$

c. The general term of the series is

$$a_j = \frac{13j-6}{j(j-2)(j+3)} = \frac{1}{j} + \frac{2}{j-2} - \frac{3}{j+3} = 3\Big(\frac{1}{j} - \frac{1}{j+3}\Big) + 2\Big(\frac{1}{j-2} - \frac{1}{j}\Big),$$

by resolving into partial fractions and rearranging. Hence, if

$$A_j = 3\Big(\frac{1}{j} + \frac{1}{j+1} + \frac{1}{j+2}\Big) + 2\Big(\frac{1}{j-2} + \frac{1}{j-1}\Big),$$

then $a_j = A_j - A_{j+1}$, for $j \geqslant 3$. Since $\lim A_j = 0$, it follows that

$$\sum_{j=3}^{\infty} \frac{13j-6}{j(j-2)(j+3)} = \lim(a_3 + a_4 + \cdots + a_{j+2}) = A_3 - \lim A_{j+3} = 3\left(\tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5}\right) + 2\left(1 + \tfrac{1}{2}\right)$$
$$= \tfrac{107}{20}.$$

Solution to Question 7. — a. If

$$a_k = \frac{\sqrt{5k^3 - 2k + 7}}{k + \sqrt[3]{3k^5 - k + 7}}, \qquad \text{and} \qquad b_k = \frac{1}{k^{1/6}},$$

then $a_k > 0$ and $b_k > 0$ if $k \geqslant 0$, and

$$\lim \frac{a_k}{b_k} = \lim \frac{k^{1/6}\sqrt{5k^3 - 2k + 7}}{k + \sqrt[3]{3k^5 - k + 7}} = \lim \frac{\sqrt{5 - 2k^{-1} + 7k^{-3}}}{k^{-2/3} + \sqrt[3]{3 - k^{-4} + 7k^{-5}}} = \frac{\sqrt{5}}{\sqrt[3]{3}},$$

which is a positive real number. Since $\sum b_k$ is a divergent p-series ($p = \frac{1}{6} \leqslant 1$), the limit comparison test implies that $\sum a_k$ is divergent.

b. If

$$a_n = \frac{n^{3n}}{(3n)!},$$

then

$$\lim \frac{a_{n+1}}{a_n} = \lim\Big\{\frac{(n+1)^{3n+3}}{(3n+3)!} \cdot \frac{(3n)!}{n^{3n}}\Big\} = \lim\Big\{\frac{(n+1)^2}{3(3n+2)(3n+1)} \cdot \Big(1 + \frac{1}{n}\Big)^{3n}\Big\} = \Big(\frac{e}{3}\Big)^3 < 1,$$

so the ratio test implies that the series $\sum a_n$ is convergent.

c. Let

$$a_\mu = \Big(1 - \frac{1}{\sqrt{\mu}}\Big)^{\mu}, \qquad \text{and} \qquad b_\mu = \frac{1}{\mu^2};$$

then $a_\mu > 0$ and $b_\mu > 0$ for $\mu > 1$, and

$$\lim \frac{a_\mu}{b_\mu} = \lim\bigg\{\Big(1 - \frac{1}{\sqrt{\mu}}\Big)^{\sqrt{\mu}} n^{2/\sqrt{\mu}}\bigg\}^{\sqrt{\mu}} = 0,$$

since $\mu^{2/\sqrt{\mu}} = e^{2(\log\mu)/\sqrt{\mu}}$ and

$$= \lim\bigg\{\Big(1 - \frac{1}{\sqrt{\mu}}\Big)^{\sqrt{\mu}} e^{2(\log\mu)/\sqrt{\mu}}\bigg\} = e^{-1} \cdot e^0 = \frac{1}{e} < 1.$$

As $\sum b_\mu$ is a convergent p-series ($p = 2 > 1$), the limit comparison test implies that the series $\sum a_\mu$ is convergent.

d. If $n \geqslant 2$ then $\csc^2(n) > 1$ and $0 < \log(n) < n$, so

$$c_n = \frac{\csc^2(n)}{\log(n)} > \frac{1}{\log(n)} > \frac{1}{n} > 0.$$

Since $\sum n^{-1}$ is a divergent p-series ($p = 1 \leqslant 1$), the comparison test implies that $\sum a_n$ is divergent.

Solution to Question 8. — a. If

$$a_k = \frac{\sin(k)(\log k)^3}{k^2 - k + 1}, \qquad \text{and} \qquad b_k = \frac{1}{k^{3/2}},$$

then, $|a_k| > 0$ and $b_k > 0$ if $k \geqslant 2$. Since $|\sin(k)| \leqslant 1$ for all $k \geqslant 1$,

$$0 \leqslant \lim \frac{|a_k|}{b_k} \leqslant \lim\Big\{\frac{(\log k)^3}{k^{1/2}} \cdot \frac{k^2}{k^2 - k + 1}\Big\} = 0$$

(so $\lim\{|a_k|/b_k\} < \infty$). Since $\sum b_k$ is a convergent p-series ($p = \frac{3}{2} > 1$), the limit comparison test implies that $\sum a_k$ is absolutely convergent.

b. If

$$a_n = \frac{n^3}{e^{\sqrt[3]{n}}}, \quad \text{and} \quad b_n = \frac{1}{n^2}, \quad \text{then } a_n, b_n > 0 \text{ if } n \geqslant 1, \text{ and} \quad \lim \frac{a_n}{b_n} = \lim \frac{n^5}{e^{\sqrt[3]{n}}} = 0.$$

Since $\sum b_n$ is a convergent p-series ($p = 2 > 1$), the limit comparison test implies that $\sum(-1)^n a_n$ is absolutely convergent.

c. If

$$c_j = \Big(\frac{2 - 3j}{2j + 5}\Big)^{j/3}, \qquad \text{then} \qquad \lim \sqrt[j]{|c_j|} = \lim\Big(\frac{3j - 2}{2j + 5}\Big)^{1/3} = \left(\tfrac{3}{2}\right)^{1/3} > 1,$$

so the series $\sum c_j$ is divergent by the root test. (Alternatively, $|c_j| \geqslant 1$ if $j \geqslant 7$, so the vanishing condition implies that $\sum c_j$ is divergent.)

d. If $k > 1$ and

$$a_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)}, \qquad \text{then} \qquad \frac{a_k}{a_{k-1}} = \frac{2k-1}{2k} < 1 \qquad \text{but} \qquad \lim \frac{a_k}{a_{k-1}} = 1,$$

so the ratio test is inconclusive. However, if $k > 1$ then

$$\frac{ka_k}{(k-1)a_{k-1}} = \frac{k(2k-1)}{2k(k-1)} = \frac{2k-1}{2k-2} > 1, \quad \text{so} \quad ka_k > (k-1)a_{k-1} > (k-2)a_{k-2} > \cdots > 1 \cdot a_1 = \tfrac{1}{2},$$

and thus $a_k > \frac{1}{2}k^{-1}$. So the comparison test implies that $\sum a_k$ is diverges with the harmonic series. On the other hand, if $k > 1$ then $0 < a_k < a_{k-1}$ (since $a_k > 0$ and $a_k/a_{k-1} < 1$), and

$$\frac{ka_k^3}{(k-1)a_{k-1}^3} = \frac{k(2k-1)^3}{(k-1)(2k)^3} = \frac{8k^3 - 12k^2 + 6k - 1}{8k^3 - 8k^2} = 1 - \frac{4k^2 - 6k + 1}{8k^3 - 8k^2} < 1,$$

so $0 < ka_k^3 < (k-1)a_{k-1}^3 < (k-2)a_{k-2}^2 < \cdots < 1 \cdot a_1^3 = \frac{1}{8}$, or $0 < a_k < \frac{1}{8}k^{-1/3}$, and hence $\lim a_k = 0$. So the alternating series test implies that $\sum(-1)^k a_k$ is convergent. Therefore, $\sum(-1)^k a_k$ is conditionally convergent. **Note**: One could instead apply the Gauß test to this series.

Solution to Question 9. — Let

$$\alpha_k = \frac{\tan(1/k)}{2^k(\log k)^{3/2}}(3x - 5)^k.$$

If $x \neq \frac{5}{3}$ then,

$$\lim\Big|\frac{\alpha_{k+1}}{\alpha_k}\Big| = \lim\Big\{\frac{\tan\big(1/(k+1)\big)|3x-5|^{k+1}}{2^{k+1}\big(\log(k+1)\big)^{3/2}} \cdot \frac{2^k(\log k)^{3/2}}{\tan(1/k)|3x-5|^k}\Big\}$$
$$= \tfrac{1}{2}|3x-5| \lim \frac{\sin\big(1/(k+1)\big)}{1/(k+1)} \cdot \frac{1/k}{\sin(1/k)} \cdot \Big(1 + \frac{1}{k}\Big)$$
$$= \tfrac{1}{2}|3x-5|.$$

The ratio test implies that $\sum \alpha_k$ is absolutely convergent if $\frac{1}{2}|3x-5|<1$, *i.e.*, $\left|x-\frac{5}{3}\right|<\frac{2}{3}$, or $1<x<\frac{7}{3}$, and divergent if $x<1$ or $x>\frac{7}{3}$. If x is 1 or $\frac{7}{3}$, then let

$$a_k = |\alpha_k| = \frac{\tan(1/k)}{(\log k)^{3/2}} \qquad \text{and} \qquad b_k = \frac{1}{k(\log k)^{3/2}};$$

then

$$\lim \frac{a_k}{b_k} = \lim \frac{\tan(1/k)}{1/k} = \lim \frac{\sin(1/k)}{1/k} = 1$$

is finite. Since $\sum b_k$ is a convergent logarithmic p-series ($p=\frac{3}{2}>1$), the series $\sum |a_k|$ is convergent by the limit comparison test. So $\sum \alpha_k$ is (absolutely) convergent if $x=1$ or if $x=\frac{7}{3}$. Therefore, $\sum \alpha_k$ has interval of convergence $\left[1,\frac{7}{3}\right]$ and radius of convergence $\frac{2}{3}$.

Solution to Question 10. — a. Writing $f(x)=\sqrt{1+3x}=\sqrt{4+3(x-1)}=2\left(1+\frac{3}{4}(x-1)\right)^{1/2}$, and using the Maclaurin expansion of $(1+t)^{1/2}$, gives

$$\begin{aligned} f(x) &= 2\left(1+\tfrac{3}{4}(x-1)\right)^{1/2} = 2\left\{1+\sum_{k=1}^{\infty}\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2}-k+1)}{k!}\left(\tfrac{3}{4}(x-1)\right)^k\right\} \\ &= 2+\sum_{k=1}^{\infty}\frac{(-1)^k 3^k(-1)\cdot 1\cdot 3\cdots(2k-3)}{2^{3k-1}k!}(x-1)^k = \sum_{k=0}^{\infty}\frac{(-1)^{k-1}3^k(2k)!}{2^{4k-1}(2k-1)(k!)^2}(x-1)^k \\ &= 2+\tfrac{3}{4}(x-1)-\tfrac{9}{64}(x-1)^2+\tfrac{27}{512}(x-1)^3-\cdots \end{aligned}$$

The Taylor series of $f(x)$ centred at 1 converges at least if $\left|\frac{3}{4}(x-1)\right|<1$ *i.e.*, $|x-1|<\frac{4}{3}$, or $-\frac{1}{3}<x<\frac{7}{3}$, and diverges if $x<-\frac{1}{3}$ or $x>\frac{7}{3}$ (since the radius of convergence of the corresponding binomial series is 1). If a_k denotes the general term of the series when $x=-\frac{1}{3}$ or when $x=\frac{7}{3}$, then

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{k-\frac{1}{2}}{k+1}.$$

Since $-\frac{1}{2}-1<-1$, the Gauß test implies that $\sum a_k$ is absolutely convergent. Therefore, the interval of convergence of the series in question is $\left[-\frac{1}{3},\frac{7}{3}\right]$.

b. Since $\sin^2(2x)=\frac{1}{2}(1-\cos 4x)$, and

$$\tfrac{1}{2}(1-\cos 4x) = \tfrac{1}{2}-\tfrac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(4x)^{2k} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}2^{4k-1}}{(2k)!}x^{2k},$$

it follows that

$$f(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}2^{4k-1}}{(2k)!}x^{2k-1}, \qquad \text{and thus} \qquad \int_0^x f(t)\,dt = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}2^{4k-2}}{k(2k)!}x^{2k}.$$

c. The interval of convergence of $g(x)$ includes $(-7,3]$, is included in $(-8,4]$, and may be equal to either of these intervals. i. Since 0 belongs to $(-7,3]$, the series $g(0)$ is convergent. ii. Since 7 does not belong to $(-8,4]$, the series $g(7)$ is divergent. iii. Since -7 belongs to $(-8,4]$ but not to $(-7,3]$, no conclusion can be drawn about the convergence of the series $g(-7)$.

Solutions to another an old Test 3

Solution to Question 1. — Separating variables and integrating gives

$$(y+1)\frac{dy}{dx}=1+\frac{1}{x}, \qquad \text{and hence} \qquad (y+1)^2=2(x+\log x)+C,$$

where the initial condition yields $(-3)^2=2+C$, or $C=7$. Solving for y (notice that $y+1<0$ because of the initial condition) then gives

$$y=-1-\sqrt{2(x+\log x)+7}.$$

Solution to Question 2. — Choose coordinates so that O is the origin and the x-intercepts of the tangents are to the left of O. If ξ is the x-intercept of the tangent line to such a curve at $P(x,y)$, where $y>0$, then

$$\frac{dx}{dy}=\frac{x-\xi}{y}, \qquad \text{or} \qquad \xi = x-y\frac{dx}{dy}.$$

Let $x=yz$, so that

$$\frac{dx}{dy}=z+y\frac{dz}{dy}, \qquad \text{and hence} \qquad \xi=-y^2\frac{dz}{dy}.$$

From $\angle XPO=\angle QPR=\angle OXP$, it follows that $|OX|=|OP|$, and therefore

$$\xi^2=x^2+y^2, \quad \textit{i.e.,} \quad y^4\left(\frac{dz}{dy}\right)^2=y^2z^2+y^2, \quad \text{or} \quad \frac{1}{\sqrt{z^2+1}}\frac{dz}{dy}=\frac{1}{y}.$$

Thus, there is a positive real number p such that $2p=e^c$ and

$$\log\left(z+\sqrt{z^2+1}\right)+c=\log y, \qquad \text{or} \qquad 2pz+2p\sqrt{z^2+1}=y.$$

Subtracting $2pz$ and squaring gives

$$4p^2=y^2-4pyz=y^2-4px, \qquad \text{or} \qquad y^2=4p(x+p).$$

Hence, every such curve is a parabola with vertical directrix and focus at O.

Solution to Question 3. — a. If $t=1/x$, then one application of l'Hôpital's Rule gives

$$\lim_{x\to\infty}x^2\log\left(\cos(1/x)\right)=\lim_{t\to 0^+}\frac{\log(\cos t)}{t^2}=\tfrac{1}{2}\lim_{t\to 0^+}\left\{\frac{\sin t}{t}\cdot\frac{1}{\cos t}\right\}=\tfrac{1}{2},$$

since $(\sin t)/t\to 1$ and $\cos t\to 1$, as $t\to 0^+$.

b. As $\lim\limits_{x\to 0}(1+x)^{1/x}=e$, the numerator and denominator each vanish as $x\to 0$. Two applications of l'Hôpital's Rule, arithmetical limit laws, and the fact that $x^{-1}\log(1+x)\to 1$ as $x\to 0$, gives

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt[x]{1+x}-e}{x} &= \lim_{x\to 0}\left\{(1+x)^{1/x}\left(\frac{1}{x(1+x)}-\frac{\log(1+x)}{x^2}\right)\right\} \\ &= e\lim_{x\to 0}\frac{x-(1+x)\log(1+x)}{x^2} = -e\lim_{x\to 0}\frac{\log(1+x)}{2x} \\ &= -\tfrac{1}{2}e. \end{aligned}$$

c. As $x\to\frac{1}{2}\pi$, $t=\sin(5x)-1\to 0$ and hence $(1+t)^{1/t}\to e$. Arithmetical properties of limits and two applications of l'Hôpital's Rule give

$$\lim_{x\to\frac{1}{2}\pi}\left\{(\sin(5x)-1)\tan^2(x)\right\}=\lim_{x\to\frac{1}{2}\pi}\frac{\sin(5x)-1}{\cos^2 x}=\lim_{x\to\frac{1}{2}\pi}\frac{5\cos(5x)}{-\sin(2x)}=\lim_{x\to\frac{1}{2}\pi}\frac{25\sin(5x)}{2\cos(2x)}=-\tfrac{25}{2},$$

and therefore $\lim\limits_{x\to\frac{1}{2}\pi}(\sin 5x)^{\tan^2 x}=\sqrt{e^{-25}}$.

Solution to Question 4. — a. First observe that

$$\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty}(b_1+b_2+\cdots+b_n) = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\log\left(\frac{n+1}{2n+1}\right) = \log\tfrac{1}{2} = -\log 2.$$

Next, the vanishing condition implies that $\sum_{n=1}^{\infty} a_n$ is divergent and that $\lim b_n = 0$.

b. i. Since

$$\frac{k^k}{k!} = \frac{k}{k}\cdot\frac{k}{k-1}\cdot\frac{k}{k-2}\cdots\frac{k}{2}\cdot\frac{k}{1} \geqslant k, \qquad \text{if} \quad k \geqslant 2,$$

it follows that $\{k^k/k!\}$ diverges to ∞.

ii. If $t = \pi^{-n}$ then

$$\lim\left\{\pi^{-n}\csc(\pi^{-n})\right\} = \lim_{t\to 0^+}\frac{t}{\sin t} = 1,$$

and if $x = \pi^{n+1}$, then one application of l'Hôpital's Rule gives

$$\lim\frac{\frac{1}{2}\pi - \operatorname{arcsec}(\pi^{n+1})}{\pi^{-n}} = \frac{1}{\pi}\lim_{x\to\infty}\frac{\frac{1}{2}\pi-\operatorname{arcsec}(x)}{x^{-1}} = \frac{1}{\pi}\lim_{x\to\infty}\frac{x}{\sqrt{x^2-1}} = \frac{1}{\pi}.$$

So

$$\lim\left\{\csc(\pi^{-n})\left(\tfrac{1}{2}\pi - \operatorname{arcsec}(\pi^n)\right)\right\} = \frac{1}{\pi},$$

which implies that the sequence $\left\{(-1)^n\csc(\pi^{-n})\left(\tfrac{1}{2}\pi - \operatorname{arcsec}(\pi^n)\right)\right\}$ is divergent.

Solution to Question 5. — a. Since $8k^2-2k-15 = (2k-3)(4k+5)$ and $8k^2-2k-3 = (2k+1)(4k-3)$, it follows that the general term of the series in question is

$$a_k = \log\left(\frac{(2k-3)(4k+5)}{(2k+1)(4k-3)}\right) = A_k - A_{k+1}, \qquad \text{where} \qquad A_k = \log\left(\frac{(2k-3)(2k-1)}{(4k-3)(4k+1)}\right),$$

for $k \geqslant 2$. Hence, the sum of the series is

$$\sum_{k=2}^{\infty} a_k = \lim(a_2+a_3+\cdots+a_{k+1}) = \lim(A_2 - A_{k+2}) = \log\left(\tfrac{1}{15}\right) - \log\left(\tfrac{1}{4}\right) = \log\left(\tfrac{4}{15}\right).$$

b. The general term of the series is

$$b_n = \tfrac{3}{8}\left(-\tfrac{3}{4}\right)^n + \frac{1}{(n+1)(n+4)} = \tfrac{3}{8}\left(-\tfrac{3}{4}\right)^n + \tfrac{1}{3}\left(\frac{1}{n+1} - \frac{1}{n+4}\right), \qquad \text{for} \quad n \geqslant 0,$$

and the sum of n terms of the series is

$$b_0 + b_1 + \cdots + b_{n-1} = \tfrac{3}{14}\left(1 - \left(-\tfrac{3}{4}\right)^n\right) + \tfrac{11}{18} - \tfrac{1}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right).$$

Therefore, the sum of the series is $\frac{3}{14} + \frac{11}{18} = \frac{52}{63}$.

Solution to Question 6. — a. If

$$a_k = 2^{2k}\left(\frac{k-3}{k}\right)^{k^2},$$

then $a_k > 0$ if $k > 3$ and

$$\lim \sqrt[k]{a_k} = \lim\left\{2^2\left(1 - \frac{3}{k}\right)^k\right\} = 4/e^3,$$

which is smaller than one, so the series $\sum a_k$ converges by the root test.

b. If $m \geqslant 1$ then $m^2+1 < 4m^2$ and $\sin(m) > -1$, so

$$b_m = \frac{\pi + e\sin(m)}{\sqrt{m^2+1}} > \frac{\pi - e}{\sqrt{4m^2}} > \tfrac{1}{2}(\pi - e)\cdot\frac{1}{m} > 0.$$

So the comparison test implies that the series $\sum b_m$ diverges with the harmonic series.

c. Let

$$c_n = \frac{2^{n-2}+5^{n+1}}{3^n + 7^{n-1}} \qquad \text{and} \qquad d_n = \left(\tfrac{5}{7}\right)^n.$$

Then $\sum d_n$ is a convergent geometric series ($|r| = \frac{5}{7} < 1$), $c_n > 0$ and $d_n > 0$ if $n \geqslant 0$, and

$$\lim\frac{c_n}{d_n} = \lim\left\{\frac{2^{n-2}+5^{n+1}}{3^n+7^{n-1}}\cdot\frac{7^n}{5^n}\right\} = \lim\frac{\frac{1}{4}\left(\frac{2}{5}\right)^n + 5}{\left(\frac{3}{7}\right)^n + \frac{1}{7}} = 35,$$

which is positive real number. Therefore, $\sum c_n$ is convergent by the limit comparison test.

d. If

$$a_n = \frac{(4n)!}{2^{4n}((2n)!)^2},$$

then

$$\frac{a_{n+1}}{a_n} = \frac{(4n+3)(4n+2)(4n+1)}{16(n+1)(2n+1)^2} = \frac{n^3 + \frac{3}{2}n^2 + \text{terms of degree} < 2}{n^3 + 2n^2 + \text{terms of degree} < 2}.$$

Since $-1 < \frac{3}{2} - 2 < 0$, the Gauß test implies that the series $\sum a_n$ is divergent.

Solution to Question 7. — a. The general term of the given series is $\cos(\pi n)a_n = (-1)^n a_n$, where

$$a_n = \frac{\operatorname{arccot}(n)}{\sqrt[3]{n}}.$$

If $b_n = n^{-4/3}$, then $a_n > 0$ and $b_n > 0$ for $n \geqslant 1$, and one application of l'Hôpital's Rule gives

$$\lim\frac{a_n}{b_n} = \lim\frac{\operatorname{arccot}(n)}{n^{-1}} = \lim\frac{n^2}{n^2+1} = 1,$$

which is a positive real number. As $\sum b_n$ is a convergent p-series ($p = \frac{4}{3} > 1$), the limit comparison test implies that $\sum a_n$ is convergent. Therefore, the series $\sum(-1)^n a_n$ is absolutely convergent.

b. If

$$a_k = \frac{\sin(1/k)}{\log k} \quad \text{and} \quad b_k = \frac{1}{k\log k}, \qquad \text{then} \qquad \lim\frac{a_k}{b_k} = \lim\frac{\sin(1/k)}{1/k} = \lim_{t\to 0^+}\frac{\sin t}{t} = 1,$$

where $t = 1/k$, and $a_k, b_k > 0$ if $k \geqslant 2$. Since $\sum b_k$ is a divergent logarithmic p-series ($p = 1$), $\sum a_k$ diverges by the limit comparison test. So $\sum(-1)^k a_k$ is not absolutely convergent. Next,

$$0 < \sin\big(1/(k+1)\big) < \sin(1/k) \qquad \text{and} \qquad \log(k+1) > \log k > 0, \qquad \text{if} \quad k \geqslant 2,$$

so $0 < a_{k+1} < a_k$. Since $\lim\{\sin(1/k)\} = 0$ and $\lim\{\log k\} = \infty$, $\lim a_k = 0$, so $\sum(-1)^k a_k$ is convergent by the alternating series test. Therefore, $\sum(-1)^k a_k$ is conditionally convergent.

c. The general term of the given series is $(-1)^k a_k$, where

$$a_k = \frac{3^{3k}(k!)^3}{(3k)!}, \qquad \text{and} \qquad \frac{a_{k+1}}{a_k} = \frac{3^3(k+1)^3}{(3k+3)(3k+2)(3k+1)} = \frac{(3k+3)(3k+3)}{(3k+2)(3k+1)} > 1,$$

so $a_{k+1} > a_k > 0$ if $k \geqslant 0$, and the Vanishing Condition implies that the series $\sum(-1)^k a_k$ is divergent.

d. If

$$a_k = \frac{1}{k^{1+1/(\log(\log k))}} = \frac{e^{-(\log k)/(\log(\log k))}}{k}, \qquad \text{and} \qquad b_k = \frac{1}{k(\log k)^2} = \frac{e^{-2(\log(\log k))}}{k},$$

then $a_k > 0$ and $b_k > 0$ if $k \geqslant 3$ and

$$\lim\frac{a_k}{b_k} = \lim e^{2(\log(\log k)) - (\log k)/(\log(\log k))} = 0,$$

since

$$\lim\left\{2(\log(\log k)) - (\log k)/(\log(\log k))\right\} = \lim_{x\to\infty}\frac{2x^2 - e^x}{x} = -\infty,$$

in which $x = \log(\log k)$). Since $\sum b_k$ is a convergent logarithmic p-series ($p = 2 > 1$), the limit comparison test implies that the series $\sum a_k$ is convergent, and therefore the series $\sum(-1)^k a_k$ is absolutely convergent.

Solution to Question 8. — If $x \neq \frac{2}{3}$ and $\alpha_k = \dfrac{(-1)^k \log k}{2^k k^{1/5}}(3x-2)^k$, then

$$\lim\left|\frac{\alpha_{k+1}}{\alpha_k}\right| = \tfrac{1}{2}|3x-2|\lim\left\{\frac{\log(k+1)}{\log k}\cdot\left(\frac{k}{k+1}\right)^{1/5}\right\} = \tfrac{1}{2}|3x-2|,$$

so the ratio test implies that the series $\sum \alpha_k$ is absolutely convergent if $\frac{1}{2}|3x-2| < 1$, *i.e.*, $0 < x < \frac{4}{3}$, and divergent if $x < 0$ or $x > \frac{4}{3}$. If $x = 0$ then $\alpha_k = k^{-1/5}\log k > k^{-1/5}$, if $k \geqslant 3$, and $\sum k^{-1/5}$ is a divergent p-series ($p = \frac{1}{5} \leqslant 1$), so $\sum \alpha_k$ is divergent by the comparison test. If $x = \frac{4}{3}$ then $\alpha_k = (-1)^k k^{-1/5}\log k$, so $\alpha_k > 0$ if $k > 1$ and $\lim \alpha_k = 0$. Also

$$\frac{d}{dk}\left\{k^{-1/5}\log k\right\} = k^{-6/5} - \tfrac{1}{5}k^{-6/5}\log k = \tfrac{1}{5}k^{-6/5}(5 - \log k),$$

which is negative if $k > e^5$, where $\{k^{-1/5}\log k\}$ is decreasing. So the Leibniz test implies that $\sum \alpha_k$ converges if $x = \frac{4}{3}$. Hence, the series $\sum \alpha_k$ has interval of convergence $\left(0, \frac{4}{3}\right]$ and radius of convergence $\frac{2}{3}$.

Solution to Question 9. — a. Since $0 < 1/(2n) < a_n$ for $n \geqslant 1$ and $\sum 1/n$ diverges, $\sum a_n$ diverges by the comparison test. On the other hand, $1/n > a_n \geqslant a_{n+1} > 1/(2n+2) > 0$, so $\{a_n\}$ is positive and nonincreasing. Since $\lim a_n = 0$, the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent. Therefore, $\sum(-1)^n a_n$ is conditionally convergent.

b. Define the sequence $\{a_n\}$ by

$$a_n = \begin{cases} \dfrac{3}{5n} & \text{if } n \text{ is odd, and} \\[2mm] \dfrac{4}{5n} & \text{if } n \text{ is even and positive.} \end{cases}$$

If $n \geqslant 1$, then

$$\frac{1}{2n} < \frac{3}{5n} \leqslant a_n \leqslant \frac{4}{5n} < \frac{1}{n}$$

If $k \geqslant 1$, then

$$a_{2k-1} = \frac{3}{5(2k-1)} \qquad \text{and} \qquad a_{2k} = \frac{4}{5\cdot 2k} = \frac{2}{5k},$$

and hence

$$\sum_{\nu=1}^{2n}(-1)^\nu a_\nu = \sum_{k=1}^{n}\left\{a_{2k} - a_{2k-1}\right\} = \sum_{k=1}^{n}\left\{\frac{2}{5k} - \frac{3}{5(2k-1)}\right\} = \sum_{k=1}^{n}\frac{k-2}{5k(2k-1)};$$

also,

$$\frac{k-2}{5k(2k-1)} > \frac{k}{5k(2k-1)} = \frac{1}{5(2k-1)} \geqslant \frac{1}{5k}.$$

Since $\sum 1/k$ diverges to ∞, it follows that

$$\lim_{n\to\infty}\sum_{\nu=1}^{2n}(-1)^\nu a_\nu = \lim_{k\to\infty}\sum_{k=1}^{n}\frac{k-2}{5k(2k-1)} = \infty,$$

which implies that $\sum(-1)^n a_n$ is divergent.

Solution to Question 10. — a. The identities $\sin(\vartheta)\cos(\vartheta) = \frac{1}{2}\sin(2\vartheta) = \frac{1}{2}\cos\left(2\vartheta - \frac{1}{2}\pi\right)$, together with the Maclaurin series of the cosine, gives

$$\begin{aligned}\sin\tfrac{1}{3}x\cos\tfrac{1}{3}x &= \tfrac{1}{2}\sin\tfrac{2}{3}x = \tfrac{1}{2}\cos\left(\tfrac{2}{3}x - \tfrac{1}{2}\pi\right) = \tfrac{1}{2}\cos\left(\tfrac{2}{3}(x - \tfrac{3}{4}\pi)\right) \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k-1}}{3^{2k}(2k)!}\left(x - \tfrac{3}{4}\pi\right)^{2k}.\end{aligned}$$

b. Since $x + 2 = 4 + (x-2)$ and $8 - 4x + x^2 = 4 + (x-2)^2$, it follows that

$$\begin{aligned}(x+2)\log(8 - 4x + x^2) &= \left(4 + (x-2)\right)\left\{\log(4) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4^n n}(x-2)^{2n}\right\} \\ &= 4\log(4) + \log(4)(x-2) + \sum_{n=1}^{\infty}\left\{\frac{(-1)^{n-1}}{4^{n-1}n}(x-2)^{2n} + \frac{(-1)^{n-1}}{4^n n}(x-2)^{2n+1}\right\}.\end{aligned}$$

c. The Maclaurin series of f is

$$5x^6\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}(3x^4)^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k\cdot 5\cdot 3^{2k+1}}{2k+1}x^{8k+10},$$

in which $f^{(2018)}(0)$ is 2018! times the coefficient corresponding to $k = 251$. Therefore,

$$f^{(2018)}(0) = -\frac{2018!\cdot 5\cdot 3^{502}}{502}.$$