Test 1, with solutions

Question 1. — a. Show that $\frac{d}{dz}\left\{(2z^2-1)\arcsin(z)+z\sqrt{1-z^2}\right\}=4z\arcsin(z)$.

b. Use the result of Part a to compute the definite integral $\int_0^{\frac12\sqrt2} z\arcsin(z)\,dz$.

c. Evaluate and simplify each of the following.

i. $\cos\big(\arctan(2/x)\big)$ ii. $\arccos\big(\sin(x)\big)$, for $\pi<x<\frac32\pi$

iii. $\lim_{y\to\infty}\operatorname{arcsec}\left(\sqrt{\frac{4y^2-1}{3y^2-2y+4}}\right)$

Solution. — a. The product and chain rules give

$$\frac{d}{dz}\left\{(2z^2-1)\arcsin(z)+z\sqrt{1-z^2}\right\}=4z\arcsin(z)+\frac{2z^2-1}{\sqrt{1-z^2}}+\sqrt{1-z^2}-\frac{z^2}{\sqrt{1-z^2}}.$$

Since

$$\sqrt{1-z^2}-\frac{z^2}{\sqrt{1-z^2}}=\frac{1-z^2}{\sqrt{1-z^2}}-\frac{z^2}{\sqrt{1-z^2}}=\frac{1-2z^2}{\sqrt{1-z^2}},$$

the result follows.

b. Using the result of Part a,

$$\int_0^{\frac12\sqrt2} z\arcsin(z)\,dz=\frac14\left\{(2z^2-1)\arcsin(z)+z\sqrt{1-z^2}\right\}\Big|_0^{\frac12\sqrt2}$$
$$=\frac14\left\{\left(0+\frac12\sqrt2\cdot\frac12\sqrt2\right)-0\right\}=\frac18.$$

c. i. If $\vartheta=\arctan(2/x)$ then $-\frac12\pi<\vartheta<\frac12\pi$ so $\sec(\vartheta)=\sqrt{\tan^2(\vartheta)+1}=\sqrt{4/x^2+1}$, and thus

$$\cos\big(\arctan(2/x)\big)=\frac{1}{\sec(\vartheta)}=\frac{1}{\sqrt{4/x^2+1}}=\frac{|x|}{\sqrt{4+x^2}}.$$

ii. If $\pi<x<\frac32\pi$, then $\arccos(\cos(x))=2\pi-x$, and so

$$\arcsin(\cos(x))=\frac12\pi-\arccos(\cos(x))=\frac12\pi-(2\pi-x)=x-\frac32\pi.$$

iii. Inspecting the dominant terms in the argument of the inverse secant gives

$$\lim_{y\to\infty}\operatorname{arcsec}\left(\sqrt{\frac{4y^2-1}{3y^2-2y+4}}\right)=\operatorname{arcsec}\left(\sqrt{\frac43}\right)=\operatorname{arcsec}\left(\frac23\sqrt3\right)=\frac16\pi.$$

Question 2. — Evaluate each of the following integrals.

a. $\int\sec(5x+7)\,dx$ b. $\int\sin(3x+8)\,dx$ c. $\int\frac{dx}{\sqrt[3]{(4x-3)^5}}$

Solution. — Each integral is standard (via an implicit affine change of variables).

a. $\int\sec(5x+7)\,dx=\frac15\log|\sec(5x+7)+\tan(5x+7)|+a$

b. $\int\sin(3x+8)\,dx=-\frac13\cos(3x+8)+b$

c. $\int\frac{dx}{\sqrt[3]{(4x-3)^5}}=-\frac{3}{8\sqrt[3]{(4x-3)^2}}+c$

Question 3. — Evaluate each of the following integrals.

a. $\int_0^9\frac{dv}{\sqrt{1+\sqrt v}}$ b. $\int\frac{1+\log(y)}{\sqrt{9-4(y\log y)^2}}\,dy$

Solution. — a. If $w=\sqrt{1+\sqrt v}$ then $v=(w^2-1)^2$, $dv=4w(w^2-1)\,dw$, $w=1$ if $v=0$ and $w=2$ if $v=3$, so

$$\int_0^9\frac{dv}{\sqrt{1+\sqrt v}}=4\int_1^2(w^2-1)\,dw=4\left(\frac13w^3-w\right)\Big|_1^2=4\left(\frac73-1\right)=\frac{16}{3}.$$

b. If $x=y\log y$ then $dx=(1+\log y)\,dy$, so

$$\int\frac{1+\log(y)}{\sqrt{9-4(y\log y)^2}}\,dy=\int\frac{dx}{\sqrt{9-4x^2}}=\frac12\arcsin\left(\frac23y\log y\right)+\beta.$$

Question 4. — Evaluate each of the following integrals.

a. $\int x^3\cos(2x)\,dx$ b. $\int_e^{e^2}\left(\frac{\log y}{y}\right)^2dy$

Solution. — a. Repeated partial integration, integrating the trigonometric factor and differentiating the power, gives

$$\int x^3\cos(2x)\,dx$$
$$=x^3\left(\frac12\sin(2x)\right)-3x^2\left(-\frac14\cos(2x)\right)+6x\left(-\frac18\sin(2x)\right)-6\left(\frac1{16}\cos(2x)\right)+\alpha$$
$$=\frac14x(2x^2-3)\sin(2x)+\frac38(2x^2-1)\cos(2x)+\alpha.$$

b. Repeated partial integration, integrating the power and differentiating the (power of the) logarithm, gives

$$\int_e^{e^2}\left(\frac{\log y}{y}\right)^2dy=-\frac{(\log y)^2}{y}\Big|_e^{e^2}+2\int_e^{e^2}\frac{\log y}{y^2}\,dy=\frac1e-\frac4{e^2}-\frac{2(\log y)}{y}\Big|_e^{e^2}+2\int_e^{e^2}\frac{dy}{y^2}$$
$$=\frac3e-\frac8{e^2}-\frac2y\Big|_e^{e^2}=\frac5e-\frac{10}{e^2}.$$

Question 5. — Evaluate each of the following integrals.

a. $\int e^{-2t}\sin(t)\,dt$ b. $\int\frac{du}{u^6\sqrt{1-4u^2}}$ c. $\int\frac{4x-3}{3x^2+x-1}\,dx$

d. $\int e^t\sqrt{1+e^{2t}}\,dt$ e. $\int\arctan\left(\frac{x}{x+2}\right)dx$

Solution. — a. Repeated partial integration (integrating the trigonometric factor and differentiating the exponential function) gives

$$\begin{aligned}\int e^{-2t}\sin(t)\,dt &= -e^{-2t}\cos(t) - 2e^{-2t}\sin(t) - 4\int e^{-2t}\sin(t)\,dt\\ &= -\tfrac{1}{5}e^{-2t}\big(\cos(t)+2\sin(t)\big)+\alpha,\end{aligned}$$

where the remaining integral in the first line is absorbed on the left side.

b. If $x = u^{-1}\sqrt{1-4u^2}$, then $x^2 = u^{-2}-4$, so $-x\,dx = u^{-3}\,du$ and $u^{-4} = (x^2+4)^2$. Hence,

$$\begin{aligned}\int\frac{du}{u^6\sqrt{1-4u^2}} &= \int\frac{1}{u^4}\cdot\frac{u}{\sqrt{1-4u^2}}\cdot\frac{du}{u^3} = \int(x^2+4)^2\cdot\frac{1}{x}\cdot(-x\,dx)\\ &= -\int(x^4+8x^2+16)\,dx\\ &= -\left\{\frac{(1-4u^2)^{5/2}}{5u^5}+\frac{8(1-4u^2)^{3/2}}{3u^3}+\frac{16\sqrt{1-4u^2}}{u}\right\}+\beta\\ &= -\frac{\sqrt{1-4u^2}}{15u^5}\Big(3(1-4u^2)^2+40(1-4u^2)+240u^4\Big)+\beta\\ &= -\frac{(128u^4+16u^2+3)\sqrt{1-4u^2}}{15u^5}+\beta.\end{aligned}$$

c. If $y = 3x^2+x-1$ then $dy = (6x+1)\,dx$ and $4x-3 = \frac{2}{3}(6x+1)-\frac{11}{3}$; also, $12y = 36x^2+12x-12 = (6x+1)^2-13$. Hence,

$$\begin{aligned}\int\frac{4x-3}{3x^2+x-1}\,dx &= \frac{2}{3}\int\frac{dy}{y}-44\int\frac{dx}{(6x+1)^2-13}\\ &= \frac{2}{3}\log|3x^2+x-1|-\frac{11}{39}\sqrt{13}\log\left|\frac{6x+1-\sqrt{13}}{6x+1+\sqrt{13}}\right|+\gamma.\end{aligned}$$

d. If $x = e^t$ then $dy = e^t\,dt$ and the integral becomes $\int\sqrt{1+x^2}\,dx$. Next, partial integration with absorption yields

$$\begin{aligned}\int\sqrt{1+x^2}\,dx &= x\sqrt{1+x^2}-\int\frac{x^2+1-1}{\sqrt{1+x^2}}\,dx = x\sqrt{1+x^2}-\int\sqrt{1+x^2}\,dx+\int\frac{dx}{\sqrt{1+x^2}}\\ &= \tfrac{1}{2}x\sqrt{1+x^2}+\tfrac{1}{2}\log\big(x+\sqrt{1+x^2}\big)+\delta.\end{aligned}$$

Therefore,

$$\int e^t\sqrt{1+e^{2t}}\,dt = \tfrac{1}{2}e^t\sqrt{1+e^{2t}}+\tfrac{1}{2}\log\big(e^t+\sqrt{1+e^{2t}}\big)+\delta.$$

e. Since

$$\frac{d}{dx}\left\{\arctan\left(\frac{x}{x+2}\right)\right\} = \frac{1}{1+\dfrac{x^2}{(x+2)^2}}\cdot\frac{2}{(x+2)^2} = \frac{1}{x^2+2x+2},$$

partial integration gives

$$\begin{aligned}\int\arctan\left(\frac{x}{x+2}\right)dx &= (x+1)\arctan\left(\frac{x}{x+2}\right)-\int\frac{x+1}{x^2+2x+2}\,dx\\ &= (x+1)\arctan\left(\frac{x}{x+2}\right)-\tfrac{1}{2}\log(x^2+2x+2)+\varepsilon.\end{aligned}$$

Bonus Question. — Attempt **EITHER** Part a **OR** or Part b.

a. Evaluate the definite integral $\displaystyle\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\frac{\vartheta}{\sin(\vartheta)}\,d\vartheta$.

b. Use partial integration to derive a reduction formula for $\int(ax^2+b)^{n/2}\,dx$ in terms of $\int(ax^2+b)^{(n-2)/2}\,dx$, and use this reduction formula to compute the integral $\int\sqrt{(3x^2-2)^5}\,dx$.

Solution. — a. If $\vartheta = \pi-\varphi$ then

$$\begin{aligned}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\frac{\vartheta}{\sin(\vartheta)}\,d\vartheta &= \int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\frac{\pi-\varphi}{\sin(\varphi)}\,d\varphi = \pi\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\csc(\varphi)\,d\varphi-\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\frac{\varphi}{\sin(\varphi)}\,d\varphi\\ &= -\tfrac{1}{2}\pi\log\big(\csc(\varphi)+\cot(\varphi)\big)\Big|_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} = -\tfrac{1}{2}\pi\log\left(\frac{\frac{2}{3}\sqrt{3}-\frac{1}{3}\sqrt{3}}{\frac{2}{3}\sqrt{3}+\frac{1}{3}\sqrt{3}}\right)\\ &= \tfrac{1}{2}\log(3).\end{aligned}$$

b. Partial integration with absorption gives

$$\begin{aligned}\int(ax^2+b)^{n/2}\,dx &= x(ax^2+b)^{n/2}-n\int(ax^2+b-b)(ax^2+b)^{(n-2)/2}\,dx\\ &= x(ax^2+b)^{n/2}+bn\int(ax^2+b)^{(n-2)/2}\,dx-n\int(ax^2+b)^{n/2}\,dx\\ &= \frac{x(ax^2+b)^{n/2}}{n+1}+\frac{bn}{n+1}\int(ax^2+b)^{(n-2)/2}\,dx.\end{aligned}$$

Applied to the given integral, this yields

$$\begin{aligned}\int(3x^2-2)^{5/2}\,dx &= \tfrac{1}{6}x(3x^2-2)^{5/2}-\tfrac{5}{3}\cdot\tfrac{1}{4}x(3x^2-2)^{3/2}+\tfrac{5}{3}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}x(3x^2-2)^{1/2}\\ &\qquad-\tfrac{5}{3}\cdot\tfrac{3}{2}\cdot 1\int\frac{dx}{\sqrt{3x^2-2}}\\ &= \tfrac{1}{12}x\Big(2(3x^2-2)^2-5(3x^2-2)+15\Big)\sqrt{3x^2-2}\\ &\qquad-\tfrac{5}{6}\sqrt{3}\log\left|\sqrt{3}\,x+\sqrt{3x^2-2}\right|+C\\ &= \tfrac{1}{4}x(6x^4-13x^2+11)\sqrt{3x^2-2}-\tfrac{5}{6}\sqrt{3}\log\left|\sqrt{3}\,x+\sqrt{3x^2-2}\right|+C.\end{aligned}$$