Test 2, with solutions

Note. — The empty ritual of appending "$+C$" to an indefinite integral will not be performed.

Question 1. — Evaluate each of the following integrals, use the variable names as given and simplify all results.

a. $\displaystyle\int \frac{x^5+x^4-7x^3+7x-1}{x^3-7x+6}\,dx$ b. $\displaystyle\int \frac{\sqrt{5x^2+3}}{x}\,dx$ c. $\displaystyle\int_0^{\frac{1}{6}\pi} \sin^2(y)\cos^4(y)\,dy$

d. $\displaystyle\int \frac{\sqrt{9+e^t}}{e^t}\,dt$ e. $\displaystyle\int \frac{du}{\sqrt[4]{1-4u^4}}$

Solution. — a. Dividing, and then resolving the integrand into partial fractions by inpsection (covering and evaluating), gives

$$\frac{x^5+x^4-7x^3+7x-1}{x^3-7x+6} = x^2+x+\frac{x^2+x-1}{(x-1)(x-2)(x+3)} = x^2+x-\frac{1}{4(x-1)}+\frac{1}{x-2}+\frac{1}{4(x+3)}.$$

Therefore,

$$\int \frac{x^5+x^4-7x^3+7x-1}{x^3-7x+6}\,dx = \tfrac{1}{3}x^3+\tfrac{1}{2}x^2+\tfrac{1}{4}\log\left|\frac{(x-2)^4(x+3)}{x-1}\right|.$$

b. If $y=\sqrt{5x^2+3}$ then $5x^2=y^2-3$ and $5x\,dx = y\,dy$, so

$$\int \frac{\sqrt{5x^2+3}}{5x^2}\cdot 5x\,dx = \int \frac{y^2}{y^2-3}\,dy = \int \left\{1+\frac{3}{y^2-3}\right\}dy = \sqrt{5x^2+3}+\tfrac{1}{2}\sqrt{3}\log\left(\frac{\sqrt{5x^2+3}-\sqrt{3}}{\sqrt{5x^2+3}+\sqrt{3}}\right).$$

c. Since $\sin^2(y)\cos^4(y) = \frac{1}{8}\sin^2(2y)(1+\cos(2y)) = \frac{1}{16}-\frac{1}{16}\cos(4y)+\frac{1}{8}\sin^2(2y)\cos(2y)$, it follows that

$$\int_0^{\frac{1}{6}\pi} \sin^2(y)\cos^4(y)\,dy = \left(\tfrac{1}{16}y-\tfrac{1}{64}\sin(4y)+\tfrac{1}{48}\sin^3(2y)\right)\Big|_0^{\frac{1}{6}\pi} = \tfrac{1}{96}\pi-\tfrac{1}{64}\cdot\tfrac{1}{2}\sqrt{3}+\tfrac{1}{48}\cdot\tfrac{3}{8}\sqrt{3} = \tfrac{1}{96}\pi.$$

d. If $x=\sqrt{9+e^t}$ then $x^2-9=e^t$ and $2x\,dx = e^t\,dt$, so partial integration gives

$$\int \frac{\sqrt{9+e^t}}{e^{2t}}\cdot e^t\,dt = \int x\cdot\frac{2x}{(x^2-9)^2}\,dx = -\frac{x}{x^2-9}+\int\frac{dx}{x^2-9}$$
$$= -\frac{\sqrt{9+e^t}}{e^t}+\tfrac{1}{6}\log\left(\frac{\sqrt{9+e^t}-3}{\sqrt{9+e^t}+3}\right).$$

e. Let $x = u^{-1}\sqrt[4]{1-4u^2}$, so $x^4+4=u^{-4}$ and $-x^3\,dx = u^{-5}\,du$. Thus,

$$\int\frac{du}{\sqrt[4]{1-4u^4}} = \int u^4\cdot\frac{u}{\sqrt[4]{1-4u^4}}\cdot\frac{du}{u^5} = -\int\frac{x^2}{x^4+4}\,dx.$$

Now $x^4+4=(x^2-2x+2)(x^2+2x+2)$, and resolving into partial fractions and integrating gives

$$\int\frac{4x^2}{x^4+4}\,dx = \int\left\{\frac{x-1+1}{x^2-2x+2}-\frac{x+1-1}{x^2+2x+2}\right\}dx = \tfrac{1}{2}\log\left(\frac{x^2-2x+2}{x^2+2x+2}\right)+\arctan(x-1)+\arctan(x+1).$$

Since $u^2(x^2\pm 2x+2) = 2u^2+\sqrt{1-4u^4}\pm 2u\sqrt[4]{1-4u^4}$, it follows that

$$\int\frac{du}{\sqrt[4]{1-4u^4}} = -\tfrac{1}{8}\log\left(\frac{2u^2+\sqrt{1-4u^4}-2u\sqrt[4]{1-4u^4}}{2u^2+\sqrt{1-4u^4}+2u\sqrt[4]{1-4u^4}}\right)$$
$$-\tfrac{1}{4}\arctan\left(\frac{\sqrt[4]{1-4u^4}-u}{u}\right)-\tfrac{1}{4}\arctan\left(\frac{\sqrt[4]{1-4u^4}+u}{u}\right).$$

Question 2. — Find the area of the region enclosed by the graphs of $y=x+12/x$ and $x+2y=18$.

Solution. — The curves meet where $2x+24/x=18-x$. Multiplying by $2x$ and subtracting the right side from the left side gives

$$0 = 3x^2-18x+24 = 3(x-2)(x-4).$$

If $0<x<2$ the difference is negative, so the line is above the curve. Therefore, the area of the region enclosed by the curves is equal to

$$\int_2^4\left(9-\tfrac{3}{2}x-12/x\right)dx = \left(9x-\tfrac{3}{4}x^2-12\log(x)\right)\Big|_2^4 = 9-12\log(2).$$

Question 3. — Evaluate each improper integral, or show that it diverges. Be sure to indicate clearly the limits involved in each improper integral. To earn full credit, display complete and well-organized solutions, and simplify all results.

a. $\displaystyle\int_0^1\frac{(\log x)^2}{\sqrt[3]{x^2}}\,dx$ b. $\displaystyle\int_{-2}^5\frac{2}{\sqrt[3]{(5x+2)^4}}\,dx$ c. $\displaystyle\int_{-\infty}^0\frac{dx}{\sqrt[3]{(x+1)^2(x-2)^4}}$

Solution. — a. Repeated partial integration gives

$$\int_0^1\frac{(\log x)^2}{\sqrt[3]{x^2}}\,dx = \lim_{t\to 0^+}3\sqrt[3]{x}(\log x)^2\Big|_t^1-6\int_0^1\frac{\log(x)}{\sqrt[3]{x^2}}\,dx = -6\lim_{t\to 0^+}\sqrt[3]{x}\log(x)\Big|_t^1+18\int_0^1\frac{dx}{\sqrt[3]{x^2}}$$
$$= 54\lim_{t\to 0^+}\sqrt[3]{x}\Big|_t^1 = 54,$$

since $\sqrt[3]{x}(\log x)^n\to 0$ as $t\to 0^+$ for any positive integer n, by basic arithmetical properties of the logarithm (dominance).

b. Since

$$\int_{-\frac{2}{5}}^5\frac{2}{\sqrt[3]{(5x+2)^4}}\,dx = \lim_{\alpha\to-\frac{2}{5}^+}\frac{-6}{5\sqrt[3]{5x+2}}\Big|_\alpha^5 = -\tfrac{2}{5}+\lim_{\alpha\to-\frac{2}{5}^+}\frac{6}{5\sqrt[3]{5\alpha+2}} = \infty,$$

it follows that the integral in question diverges.

c. If $y^3=(x+1)/(x-2)$, then

$$x = \frac{2y^3+1}{y^3-1} = 2+\frac{3}{y^3-1},\quad dx = -\frac{9y^2}{y^3-1)^2}\,dy,\quad x-2=\frac{3}{y^3-1}\quad\text{and}\quad x+1=\frac{3y^3}{y^3-1},$$

so that

$$\int\frac{dx}{\sqrt[3]{(x+1)^2(x-2)^4}} = \int\frac{1}{(x+1)(x-2)}\sqrt[3]{\frac{x+1}{x-2}}\,dx = -\int dy = -\sqrt[3]{\frac{x+1}{x-2}}.$$

The integral in question is improper because the lower limit is infinite and because the integrand has an infinite discontinuity at -1, so

$$\int_{-\infty}^0\frac{dx}{\sqrt[3]{(x+1)^2(x-2)^4}} = -\lim_{t\to-\infty}\sqrt[3]{\frac{x+1}{x-2}}\Big|_t^{-2}-\lim_{t\to-1^-}\sqrt[3]{\frac{x+1}{x-2}}\Big|_{-2}^t-\lim_{t\to-1^+}\sqrt[3]{\frac{x+1}{x-2}}\Big|_t^0$$
$$= 1+2^{-1/3}.$$

Question 4. — Let $\mathscr{R}$ denote the region enclosed by the graphs of $y=3xe^{-x}$ and $y=\frac{1}{3}x$.

a. Find the exact coordinates of all points of intersection of the curves defined by $y=3xe^{-x}$ and $y=\frac{1}{3}x$.

Write an integral which represents the volume of:

b. the solid generated by revolving $\mathscr{R}$ about the x axis;

c. the solid generated by revolving $\mathscr{R}$ about the line $x=3$;

d. the solid generated by revolving $\mathscr{R}$ about the line $y = 2$;

e. the solid with base $\mathscr{R}$, and whose cross sections perpendicular to the x axis are equilateral triangles.

Solution. — a. The curves meet where $0 = 3xe^{-x} - \frac{1}{3}x = \frac{1}{3}x(9e^{-x} - 1)$, *i.e.*, where $x = 0$ and $y = 0$, and where $x = \log 9 = 2\log 3$ and $y = \frac{2}{3}\log 3$. If $0 < x < 2\log(3)$ then $e^x < 9$, so the curve is above the line between their points of intersection.

b. The solid consists of annuli of inner radius $\frac{1}{3}x$ and outer radius $3xe^{-x}$, for $0 \leqslant x \leqslant 2\log 3$, so its volume is

$$\pi \int_0^{2\log 3} \left\{ (9x^2 e^{-2x} - \tfrac{1}{9}x^4 \right\} dx.$$

c. The solid consists of cylindrical shells of radius $3 - x$ and height $3xe^{-x} - \frac{1}{3}x$, for $0 \leqslant x \leqslant 2\log x$, so its volume is

$$2\pi \int_0^{2\log 3} (3 - x)(3xe^{-x} - \tfrac{1}{3}x)\, dx.$$

d. The solid consists of annuli of inner radius $2 - 3xe^{-x}$ and outer radius $2 - \frac{1}{3}x$, for $0 \leqslant x \leqslant 2\log x$, so its volume is

$$\pi \int_0^{2\log 3} \left\{ (2 - \tfrac{1}{3}x)^2 - (2 - 3xe^{-x})^2 \right\} dx.$$

e. The solid is composed of equilateral triangles each of whose base $3xe^{-x} - \frac{1}{3}x$ and height $\frac{1}{2}\sqrt{3}(3xe^{-x} - \frac{1}{3}x)$, for $0 \leqslant x \leqslant 2\log 3$, so its volume is

$$\tfrac{1}{4}\sqrt{3} \int_0^{2\log 3} (3xe^{-x} - \tfrac{1}{3}x)^2\, dx.$$

Question 5. — Compute the length of the curve defined by $y = \ln\left(\dfrac{e^x + 1}{e^x - 1}\right)$, for $\log 2 \leqslant x \leqslant \log 4$.

Solution. — Since

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-2e^x}{e^{2x} - 1}\right)^2 = \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right)^2 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2,$$

the length of the curve is equal to

$$\int_{\log 2}^{\log 4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\log 2}^{\log 4} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \log(e^x - e^{-x})\Big|_{\log 2}^{\log 4} = \log\left(\frac{4 - \frac{1}{4}}{2 - \frac{1}{2}}\right) = \log\left(\tfrac{5}{2}\right).$$

Question 6. — Show that $\displaystyle\int_0^1 \frac{\log(x + 1)}{\sqrt[3]{x^5}\arctan(x)}\, dx$ diverges, and $\displaystyle\int_1^\infty \frac{\log(x + 1)}{\sqrt[3]{x^5}\arctan(x)}\, dx$ converges.

Solution. — The inequality $(t - 1)/t < \log(t) < t - 1$, for $t > 1$, implies that $\frac{1}{2}x < \log(x + 1) < x$ if $0 < x < 1$, and the inequality $\vartheta\cos(\vartheta) < \sin(\vartheta) < \vartheta$, for $0 < \vartheta < \frac{1}{2}\pi$, implies that $\vartheta < \tan(\vartheta) < 2\vartheta$ if $0 < \vartheta < \frac{1}{3}\pi$, and hence that $\frac{1}{2}x < \arctan(x) < x$ if $0 < x < \sqrt{3}$. Therefore,

$$\frac{\log(x + 1)}{\sqrt[3]{x^5}\arctan(x)} > \frac{\frac{1}{2}x}{\sqrt[3]{x^5} \cdot x} = \frac{1}{2x} > 0, \quad \text{for} \quad 0 < x < 1.$$

Since $\displaystyle\int_0^1 \frac{dx}{x}$ is a divergent improper integral ($p = 1$ in the scale of powers at zero), it follows that the improper integral $\displaystyle\int_0^1 \frac{\log(x + 1)}{\sqrt[3]{x^5}\arctan(x)}\, dx$ is divergent.

The inequality $(x - 1)/x < \log(x) < x - 1$, for $x > 1$, implies that $0 < \frac{1}{3}\log(x) = \log\left(x^{1/3}\right) < x^{1/3}$, or $0 < \log(x) < 3x^{1/3}$, if $x > 1$. Also, $\arctan(x) > \frac{1}{3}\pi > 1$ if $x > \sqrt{3}$, and thus

$$0 < \frac{\log(x + 1)}{\sqrt[3]{x^5}\arctan(x)} < \frac{3x^{1/3}}{\sqrt[3]{x^5 \cdot 1}} = \frac{3}{x^{4/3}}, \quad \text{for} \quad x > \sqrt{3}.$$

Since $\displaystyle\int_1^\infty \frac{dx}{x^{4/3}}$ is a convergent improper integral ($p = \frac{4}{3} > 1$ in the scale of powers at ∞), it follows that the improper integral $\displaystyle\int_1^\infty \frac{\log(x + 1)}{\sqrt[3]{x^5}\arctan(x)}\, dx$ is convergent.

Bonus Question. — If possible, give an example of a function f which is continuous on $[0, \infty)$, and satisfies

- $f(x) \geqslant 0$ if $x \geqslant 0$,
- $f(n) = n^n$ for every positive integer n and
- the improper integral $\displaystyle\int_0^\infty f(x)\, dx$ is convergent.

If there is no such function, explain clearly why not.

Solution. — Define f by

$$f(x) = \begin{cases} n^n - \left(2n^2\right)^n |x - n| & \text{if } n \text{ is a positive integer and } |x - n| < (2n)^{-n}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If n is a positive integer then $f(n) = n^n$, $f(x) > 0$ if $|x - n| < (2n)^{-n}$ and

$$f(x) \to n^n - \left(2n^2\right)^n (2n)^{-n} = n^n - n^n = 0 \quad \text{as} \quad x \to n \pm (2n)^{-n\mp}.$$

In particular, f is continuous and non-negative on $\mathbb{R}$. The improper integral of f on $[0, \infty)$ is the sum of the integrals

$$\int_{n-(2n)^{-n}}^{n+(2n)^{-n}} f(x)\, dx = \tfrac{1}{2} \cdot n^n \cdot 2(2n)^{-n} = \left(\tfrac{1}{2}\right)^n, \qquad \text{for} \quad n = 1, 2, 3, \&c., \qquad (*)$$

since each integral represents the area of a triangle of base $2(2n)^{-n}$ and height n^n. From the factorization of a difference of powers, $1 - x^n = (1 - x)(1 + x + x^2 + \cdots + x^{n-1})$, it follows that

$$1 - \left(\tfrac{1}{2}\right)^n = \left(1 - \tfrac{1}{2}\right)\left(1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{2}\right)^{n-1}\right) = \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots + \left(\tfrac{1}{2}\right)^n.$$

Thus, letting $n \to \infty$ and using $(*)$ gives

$$\int_0^\infty f(x)\, dx = 1.$$