Notes

Below is a sample test. It indicates the format of the test, but naturally it will not be representative of the actual questions. To review for the test you should study the exercises on the first and second review sheets, and only after you have done this attempt the questions below. For the purpose of timing, work Questions 1–5, and **one** of the remaining four questions.

When you write solutions, keep the following in mind.

– Keep all rough work separate from the main flow of a solution.

– Use correct notation and terminology at all times. Do not leave out the differential (*e.g.*, dx) from any integrals.

– Never introduce a superfluous change of variables, especially not an affine change of variables. Also, integrals such as $\int (2x-3)(x^2-3x+5)^{-2}\,dx$ and $\int 5(x^2+x+1)^{-1/2}\,dx$ should be evaluated by inspection, not by an explicit change of variables.

– Resolve proper rationals expressions into partial fractions efficiently. Do not write out equations and solve for coefficients if a denominator has only linear factors, none of which is repeated.

– Simplify intermediate results and final answers (combine logarithms, combine fractional powers of binomials, simplify fractions, *etc.*).

– Never write garbage such as $\arctan(\infty)$, $e^{-5/0^+}$, *etc.*. Doing so will cost you marks.

– Never bother to write expressions such as $\tan(\arcsin(\dots))$ or $\sin(2\operatorname{arcsec}(\dots))$ as part of a solution. Such expressions are worth no more than the expressions, *e.g.*, $\tan(\vartheta)$ or $\sin(2\vartheta)$ which precede them.

An old Test 2

Question 1. — Evaluate each of the following integrals. To earn full credit simplify your final answers.

a. $\displaystyle\int \frac{4x^4-12x^3-31x^2+60x+30}{2x^3-7x^2-12x+45}\,dx$ b. $\displaystyle\int \frac{2x^4+7x^3+21x^2+20x+5}{(x+2)(x^2+2x+5)^2}\,dx$

c. $\displaystyle\int \frac{\arccos(\sqrt{x})}{\sqrt{x^3-x^4}}\,dx$ d. $\displaystyle\int \frac{dx}{x^2(x^{5/3}+1)^{7/5}}$ e. $\displaystyle\int_1^{e^{\pi/6}} \frac{\sin^4(\log y)\cos^2(\log y)}{y}\,dy$

Question 2. — For each improper integral, evaluate it or else show that it diverges. To earn full credit, display complete and well-organized solutions, and simplify your final answers.

a. $\displaystyle\int_1^{\infty} \frac{x-10}{6x^3+11x^2-4x-4}\,dx$ b. $\displaystyle\int_0^1 \frac{\log(x)}{\sqrt{1-x}}\,dx$ c. $\displaystyle\int_{e^4}^{\infty} \frac{\sqrt{\log(x)-1}}{x(\log x)^2}\,dx$

Question 3. — Find the area of the region enclosed by the curve defined by $y(x^2+1)=x$ and its tangent line at the point where $x=3$.

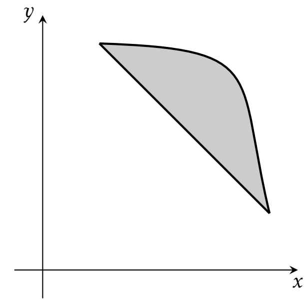
Question 4. — Let $\mathscr{S}$ be the region enclosed by the graphs of $y=x^2$ and $y=x^2e^{2-x}$. Write an integral which is equal to the volume of

a. the solid obtained by revolving $\mathscr{S}$ about: the x axis; the y axis; the line defined by $y=5$; the line defined by $x=3$.

b. the solid whose base is $\mathscr{S}$ and whose cross sections perpendicular to the x axis are semicircles.

Question 5. — Compute the length of the curve $\{(x,y)\colon y=\frac{1}{2}x^2-\frac{1}{4}\log(x),\ 1\leqslant x\leqslant 9\}$.

Question 6. — Below is a sketch of the graph of a function f on the interval $[1,4]$, together with the line segment joining the points $(1,4)$ and $(4,1)$. The shaded region is not symmetric; its area is equal to 3.



a. Find $\displaystyle\int_1^4 f^{-1}(x)\,dx$.

b. Let $\mathscr{R}$ be the region between the graph of the derivative f' of f and the x axis, for $1\leqslant x\leqslant 4$. Compute the volume of the solid obtained by revolving $\mathscr{R}$ about the line defined by $x=-1$.

Question 7. — Determine whether the improper integral

$$\int_0^{\infty} \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}}\,dx$$

is converent or divergent. To earn full credit, write a complete and well-organized solution, and justify all claims precisely and carefully.

Question 8. — By deriving a reduction formula, show that

$$\lim_{k\to\infty}\frac{\mathscr{W}_k}{\mathscr{W}_{k-1}}=1, \qquad \text{where} \qquad \mathscr{W}_k=\int_0^1 (1-x^2)^{k/2}\,dx.$$

Use this result to derive a limit formula for π.

Question 9. — A right circular cylinder of radius r and height h is cut into two parts by a plane which meets the top of the cylinder on its circumference and meets the base of the cylinder in a diameter. Compute the volume of the smaller part.

Solution to Question 1. — a. Polynomial division gives

$$\frac{4x^4-12x^3-31x^2+60x+30}{2x^3-7x^2-12x+45}=2x+1-\frac{18x+15}{2x^3-7x^2-12x+45}.$$

To factorize the denominator, notice that it vanishes at 3, so it is divisible by $x-3$. Using this to factorize the denominator by inspection gives

$$2x^3-7x^2-12x+45=(x-3)(2x^2-x-15)=(x-3)^2(2x+5).$$

The second and third coefficients in the resolution

$$\frac{18x+15}{(x-3)^2(2x+5)}=\frac{a}{x-3}+\frac{69}{11(x-3)^2}-\frac{120}{121(2x+5)},$$

are found by covering and evaluating, and then clearing denominators and comparing the quadratic coefficients gives $2a+\frac{120}{121}=0$, or $a=-\frac{60}{121}$. Hence,

$$\begin{aligned}&\int\frac{4x^4-12x^3-31x^2+60x+30}{2x^3-7x^2-12x+45}\,dx\\&=\int\left\{2x+1-\frac{60}{121(x-3)}-\frac{69}{11(x-3)^2}+\frac{120}{121(2x+5)}\right\}dx\\&=x^2+x+\frac{69}{11(x-3)}+\frac{60}{121}\log\left|\frac{2x+5}{x-3}\right|+\alpha.\end{aligned}$$

b. The first coefficient in the resolution

$$\frac{1}{x+2}+\frac{ax+b}{x^2+2x+5}+\frac{cx+d}{(x^2+2x+5)^2}=\frac{2x^4+7x^3+21x^2+20x+5}{(x+2)(x^2+2x+5)^2},$$

is found by inspection (covering and evaluating). Clearing denominators gives

$$2x^4+7x^3+21x^2+20x+5=(x^2+2x+5)^2+(ax+b)(x+2)(x^2+2x+5)+(cx+d)(x+2).$$

Comparing the coefficients of x^4 down to x gives, $a+1=2$, so $a=1$, then $4+4a+b=7$, so $b=-1$, then $14+9a+4b+c=21$, so $c=2$, and finally $20+10a+9b+2c+d=20$, so $d=-5$. Next,

$$\begin{aligned}\int\frac{x-1}{x^2+2x+5}\,dx&=\int\frac{x+1}{x^2+2x+5}\,dx-2\int\frac{dx}{(x+1)^2+4}\\&=\tfrac{1}{2}\log(x^2+2x+5)-\arctan\left(\tfrac{1}{2}(x+1)\right),\end{aligned}$$

and likewise

$$\int\frac{2x-5}{(x^2+2x+5)^2}\,dx=-\frac{1}{x^2+2x+5}-7\int\frac{dx}{(x^2+2x+5)^2}$$

.

If $t=x+1$ then $x^2+2x+5=t^2+4$; adding and subtracting t^2 in the numerator and then integrating by parts gives

$$\begin{aligned}\int\frac{4\,dt}{(t^2+4)^2}&=\int\frac{dt}{t^2+4}-\int t\cdot\frac{t\,dt}{(t^2+4)^2}=\int\frac{dt}{t^2+4}+\frac{t}{2(t^2+4)}-\frac{1}{2}\int\frac{dt}{t^2+4}\\&=\frac{x+1}{2(x^2+2x+5)}+\tfrac{1}{4}\arctan\left(\tfrac{1}{2}(x+1)\right);\end{aligned}$$

therefore,

$$\begin{aligned}\int\frac{2x-5}{(x^2+2x+5)^2}\,dx&=-\frac{1}{x^2+2x+5}-\frac{7(x+1)}{8(x^2+2x+5)}-\frac{7}{16}\arctan\left(\tfrac{1}{2}(x+1)\right)\\&=-\frac{7x+15}{8(x^2+2x+5)}-\frac{7}{16}\arctan\left(\tfrac{1}{2}(x+1)\right).\end{aligned}$$

Combining these results, the integral in question is equal to

$$-\frac{7x+15}{8(x^2+2x+5)}-\frac{23}{16}\arctan\left(\tfrac{1}{2}(x+1)\right)+\tfrac{1}{2}\log\left|(x+2)^2(x^2+2x+5)\right|+\beta,$$

since $\log|x+2|$ is a primitive function of the first term in the resolution.

c. If $\psi=\arccos(\sqrt{x})$, then $x=\cos^2(\psi)$, $dx=-2\cos(\psi)\sin(\psi)\,d\psi$ and $\sqrt{1-x}=\sin(\psi)$, so

$$\int\frac{\arccos(\sqrt{x})}{\sqrt{x^3-x^4}}\,dx=\int\frac{\arccos(\sqrt{x})}{x^{3/2}\sqrt{1-x}}\,dx=\int\frac{\psi(-2\cos(\psi)\sin(\psi))}{\cos^3(\psi)\sin(\psi)}\,d\psi=-2\int\psi\sec^2(\psi)\,d\psi.$$

Partial integration gives

$$\int\psi\sec^2(\psi)\,d\psi=\psi\tan(\psi)-\int\tan(\psi)\,d\psi=\psi\tan(\psi)+\log\left|\cos(\psi)\right|.$$

Therefore,

$$\int\frac{\arccos(\sqrt{x})}{\sqrt{x^3-x^4}}\,dx=-2\arccos(\sqrt{x})\sqrt{\frac{1-x}{x}}-\log(x)+\gamma.$$

d. If $y=x^{-1/3}(x^{5/3}+1)^{1/5}$ then $y^5=1+x^{-5/3}$, and $-3y^4\,dy=x^{-8/3}\,dx$. The integral is equal to

$$\int x^{-5/3}\cdot\left(\frac{x^{1/3}}{(x^{5/3}+1)^{1/5}}\right)^7\cdot\frac{dx}{x^{8/3}}=-3\int\left(y^2-y^{-3}\right)dy=-y^3-\tfrac{3}{2}y^{-2}+G=-\frac{2y^5+3}{2y^2}.$$

Since $2y^5+3=5+2x^{-5/3}=x^{-5/3}(5x^{5/3}+2)$, it follows that

$$\int\frac{dx}{x^2(x^{5/3}+1)^{7/5}}=\frac{5x^{5/3}+2}{2x^{5/3}}\cdot\frac{x^{2/3}}{(x^{5/3}+1)^{2/5}}+\delta=-\frac{5x^{5/3}+2}{2x(x^{5/3}+1)^{2/5}}+\delta.$$

e. The duplication identities for the sine and cosine give

$$\sin^4(\vartheta)\cos^2(\vartheta)=\tfrac{1}{8}\left(\sin^2(2\vartheta)-\sin^2(2\vartheta)\cos(2\vartheta)\right)=\tfrac{1}{16}-\tfrac{1}{16}\cos(4\vartheta)-\tfrac{1}{8}\sin^2(2\vartheta)\cos(2\vartheta).$$

If $\vartheta=\log(y)$ then $d\vartheta=y^{-1}\,dy$, so the integral is equal to

$$\begin{aligned}\int_0^{\frac{1}{6}\pi}\sin^2(\vartheta)\cos^4(\vartheta)\,d\vartheta&=\left(\tfrac{1}{16}\vartheta-\tfrac{1}{64}\sin(4\vartheta)-\tfrac{1}{48}\sin^3(2\vartheta)\right)\Bigg|_0^{\frac{1}{6}\pi}=\tfrac{1}{16}\cdot\tfrac{1}{6}\pi-\tfrac{1}{6}4\cdot\tfrac{1}{2}\sqrt{3}-\tfrac{1}{4}8\cdot\tfrac{3}{8}\sqrt{3}\\&=\tfrac{1}{96}\pi-\tfrac{1}{64}\sqrt{3}.\end{aligned}$$

Solution to Question 2. — a. The denominator of the integrand vanishes if $x=-2$, so it is divisible by $x+2$, and factorizing gives $6x^3+11x^2-4x-4=(x+2)(6x^2-x-2)=(x+2)(2x+1)(3x-2)$. Resolving the integrand into partial fractions by inspection (covering and evaluating) yields

$$\frac{x-10}{(x+2)(2x+1)(3x-2)}=-\frac{1}{2(x+2)}+\frac{2}{2x+1}-\frac{3}{2(3x-2)}.$$

Therefore,

$$\int_1^\infty\frac{x-10}{6x^3+11x^2-4x-4}\,dx=\tfrac{1}{2}\lim_{\alpha\to\infty}\log\left(\frac{(2x+1)^2}{(x+2)(3x-2)}\right)\Bigg|_1^\alpha=\tfrac{1}{2}\log\left(\tfrac{4}{3}\cdot\tfrac{1}{3}\right)=\log\left(\tfrac{2}{3}\right).$$

b. If $y=\sqrt{1-x}$, then $x=1-y^2$, $dx=-2y\,dy$, and hence

$$\int_0^1\frac{\log(x)}{\sqrt{1-x}}\,dx=2\int_0^1\log(1-y^2)\,dy,$$

where the integrand on the right has a removable discontinuity at the origin, and the impropriety of the integral occurs only as $y \to 1^-$. Partial integration, and then integrating by inspection, gives

$$\begin{aligned}\int \log(1-y^2)\,dy &= y\log(1-y^2) + 2\int \frac{y^2 - 1 + 1}{1-y^2}\,dy = y\log(1-y^2) - 2y - \log\Big(\frac{1-y}{1+y}\Big)\\ &= (y-1)\log(1-y) + (y+1)\log(1+y) - 2y,\end{aligned}$$

for $-1 < y < 1$. Recall that if $\beta > 0$ then $z^\beta(-\log z)^\alpha \to 0$ as $z \to 0^+$, by elementary properties of the logarithm, so $(t-1)\log(1-t) \to 0$ as $t \to 1^-$. Therefore,

$$\int_0^1 \frac{\log(x)}{\sqrt{1-x}}\,dx = 2\lim_{t\to 1^-}\Big\{(t-1)\log(1-t) + (t+1)\log(1+t) - 2t\Big\} = 2(2\log 2 - 2) = 4\log(2e^{-1}).$$

c. If $y = \sqrt{\log(x) - 1}$ then $y^2 + 1 = \ln(x)$ and $2y\,dy = x^{-1}\,dx$. Partial integration then gives

$$\begin{aligned}\int_{e^4}^{\infty} \frac{\sqrt{\log(x)-1}}{x(\log x)^2}\,dx &= \int_{\sqrt{3}}^{\infty} y\cdot\frac{2y}{(y^2+1)^2}\,dy = -\lim_{\alpha\to\infty} \frac{y}{y^2+1}\bigg|_{\sqrt{3}}^{\infty} + \int_{\sqrt{3}}^{\infty}\frac{dy}{y^2+1}\\ &= \tfrac{1}{4}\sqrt{3} + \lim_{\alpha\to\infty}\arctan(y)\bigg|_{\sqrt{3}}^{\infty} = \tfrac{1}{4}\sqrt{3} + \tfrac{1}{6}\pi\end{aligned}$$

Solution to Question 3. — Since

$$y\bigg|_{x=3} = \frac{x}{x^2+1}\bigg|_{x=3} = \frac{3}{10} \qquad\text{and}\qquad \frac{dy}{dx}\bigg|_{x=3} = \frac{1-x^2}{(1+x^2)^2}\bigg|_{x=3} = -\frac{2}{25},$$

the tangent line to the curve is defined by $y = \frac{3}{10} - \frac{2}{25}(x-3) = -\frac{2}{25}x + \frac{27}{50}$. Let $\overline{y}$ denote the y coordinate on the curve and $\underline{y}$ the y coordinate on the tangent line, then

$$\frac{x}{x^2+1} + \frac{2}{25}x - \frac{27}{50} = \frac{4x^3 - 27x^2 + 54x - 27}{50(x^2+1)} = \frac{(x-3)^2(4x-3)}{50(x^2+1)},$$

where the square factor due to the tangency and the other factor is obtained by inspection. Thus, $\overline{y} - \underline{y}$ is positive if $\frac{3}{4} < x < 3$, so the area between the curve and its tangent line is equal to

$$\begin{aligned}\int_{3/4}^{3}\Big\{\frac{x}{x^2+1} + \frac{2}{25}x - \frac{27}{50}\Big\}dx &= \Big\{\tfrac{1}{2}\log(x^2+1) + \tfrac{1}{25}x^2 - \tfrac{27}{50}x\Big\}\bigg|_{3/4}^{3} = \tfrac{1}{2}\log\big(\tfrac{32}{5}\big) + \tfrac{1}{25}\cdot\tfrac{135}{16} - \tfrac{27}{50}\cdot\tfrac{9}{4}\\ &= \tfrac{1}{2}\log\big(\tfrac{32}{5}\big) - \tfrac{351}{400}.\end{aligned}$$

Solution to Question 4. — The curves meet where $x^2e^{2-x} = x^2$, equivalently $x^2(e^{2-x}-1) = 0$, whose solutions are 0 and 2. If $0 < x < 2$ then $e^{2-x} > 1$, so $x^2 < x^2e^{2-x}$.

a. The solid obtained by revolving $\mathscr{S}$ about the x axis consists of annuli of inner radius x^2 and outer radius x^2e^{2-x}, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$\pi\int_0^2\Big\{(x^2e^{2-x})^2 - x^4\Big\}dx.$$

The solid obtained by revolving $\mathscr{S}$ about the y axis consists of concentric cylindrical shells of radius x and height $x^2(e^{2-x}-1)$, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$2\pi\int_0^2 x^3(e^{2-x}-1)\,dx.$$

The solid obtained by revolving $\mathscr{S}$ about the line defined by $y = 5$ consists of annuli of inner radius $5 - x^2e^{2-x}$ and outer radius $5 - x^2$, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$\pi\int_0^2\Big\{(5-x^2)^2 - (5-x^2e^{2-x})^2\Big\}dx.$$

The solid obtained by revolving $\mathscr{S}$ about the line defined by $x = 3$ consists of concentric cylindrical shells of radius $3 - x$ and height $x^2(e^{2-x}-1)$, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$2\pi\int_0^2 (3-x)x^2(e^{2-x}-1)\,dx.$$

b. Cross sections of the solid perpendicular to the x axis are semicircles of radius $x^2(e^{2-x}-1)$, and area $\frac{1}{2}\pi x^4(e^{2-x}-1)^2$, for $0 \leqslant x \leqslant 2$, so its volume is equal to

$$\tfrac{1}{2}\pi\int_0^2 x^4(e^{2-x}-1)^2\,dx.$$

Solution to Question 5. — If $y = \frac{1}{2}x^2 - \frac{1}{4}\log(x)$ then $\frac{dy}{dx} = x - \frac{1}{4}x^{-1}$, so

$$1 + \Big(\frac{dy}{dx}\Big)^2 = 1 + x^2 - \tfrac{1}{2} + \tfrac{1}{16}x^{-2} = x^2 + \tfrac{1}{2} + \tfrac{1}{16}x^{-2} = \big(x + \tfrac{1}{4}x^{-1}\big)^2,$$

which is positive if $1 \leqslant x \leqslant 9$. So the length of the curve is

$$\int_1^9\big(x + \tfrac{1}{4}x^{-1}\big)dx = \big(\tfrac{1}{2}x^2 + \tfrac{1}{4}\log(x)\big)\bigg|_1^9 = 40 + \tfrac{1}{2}\log(3).$$

Solution to Question 6. — a. The integral of f^{-1} on the interval $[1,4]$ is equal to the sum of the area of the shaded region and the area of the trapezoid to the left of the shaded region and to the right of the y axis. The trapezoid has base 3 and heights 1 and 4, so its area is $\frac{1}{2}(1+4)\cdot 3 = \frac{15}{2}$. Therefore,

$$\int_1^4 f^{-1}(x)\,dx = \tfrac{15}{2} + 3 = \tfrac{21}{2}.$$

b. The solid consists of concentric cylindrical shells of radius $x+1$ and height $-f'(x)$, for $1 \leqslant x \leqslant 4$, so its volume is equal to

$$-2\pi\int_1^4 (x+1)f'(x)\,dx = -2\pi(x+1)f(x)\bigg|_1^4 + 2\pi\int_1^4 f(x)\,dx,$$

via partial integration. The first term is $-2\pi(5-8) = 6\pi$ and the second term is $2\pi\cdot\frac{21}{2} = 21\pi$ as in part a. Therefore, the solid has volume 27π.

Solution to Question 7. — The inequality $\vartheta\cos(\vartheta) < \sin(\vartheta) < \vartheta$, which is valid at least if $0 < \vartheta < \frac{1}{2}\pi$, implies that if $0 < x < \frac{1}{6}\pi$, then $0 < 1-\cos(3x) = 2\sin^2\left(\frac{3}{2}x\right) < \frac{9}{2}x^2$, and thus

$$0 < \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}} < \sqrt[3]{\frac{9}{2}}\cdot\frac{x^{2/3}}{x^{3/2}} = \sqrt[3]{\frac{9}{2}}\cdot\frac{1}{x^{5/6}}.$$

Since $\int_0^1 \frac{dx}{x^{5/6}}$ is a convergent improper integral ($p = \frac{5}{6} < 1$ in the scale of powers near zero), it follows that the improper integral

$$\int_0^1 \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}}\,dx$$

is convergent. Next, since $-1 \leqslant \cos(3x) \leqslant 1$ for all real values of x, it follows that $0 \leqslant 1-\cos(3x) \leqslant 2$, and hence

$$0 < \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}} < \frac{\sqrt[3]{2}}{x^{3/2}} \qquad \text{if} \quad x > 1.$$

Since $\int_1^\infty \frac{dx}{x^{3/2}}$ is a convergent improper integral ($p = \frac{3}{2} > 1$ in the scale of powers at ∞), it follows that the improper integral

$$\int_1^\infty \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}}\,dx$$

is convergent. Therefore, the improper integral

$$\int_0^\infty \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}}\,dx = \int_0^1 \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}}\,dx + \int_1^\infty \frac{\sqrt[3]{1-\cos(3x)}}{x\sqrt{x}}\,dx$$

is convergent.

Solution to Question 8. — Partial integration and rearranging gives

$$\begin{aligned}\mathscr{W}_k = \int_0^1 (1-x^2)^{k/2}\,dx &= x(1-x^2)^{k/2}\Big|_0^1 + k\int_0^1 x^2(1-x^2)^{(k-2)/2}\,dx \\ &= -k\int_0^1 (1-x^2)^{k/2}\,dx + k\int_0^1 (1-x^2)^{(k-2)/2}\,dx \\ &= \frac{k}{k+1}\int_0^1 (1-x^2)^{(k-2)/2}\,dx,\end{aligned}$$

or

$$\mathscr{W}_k = \frac{k}{k+1}\mathscr{W}_{k-2},$$

provided $k > 0$ (so that the integral $\mathscr{W}_{k-2}$ is convergent). If $0 < k < k'$ and $0 < x < 1$, then

$$0 < (1-x^2)^{k'/2} < (1-x^2)^{k/2}, \qquad \text{and hence} \qquad 0 < \mathscr{W}_{k'} < \mathscr{W}_k,$$

by the monotonicity of the definite integral. Thus, if $k > 2$ then

$$0 < \mathscr{W}_k < \mathscr{W}_{k-1} < \mathscr{W}_{k-2} = \frac{k+1}{k}\mathscr{W}_k,$$

and hence

$$\frac{k}{k+1} < \frac{\mathscr{W}_k}{\mathscr{W}_{k-1}} < 1, \qquad \text{so} \qquad \lim_{k\to\infty}\frac{\mathscr{W}_k}{\mathscr{W}_{k-1}} = 1.$$

Notice that

$$\mathscr{W}_0 = 1, \qquad \text{and} \qquad \mathscr{W}_{-1} = \frac{1}{2}\pi,$$

so this last limit is

$$\lim_{n\to\infty}\left\{\frac{1}{\frac{\pi}{2}}\cdot\frac{\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}}{\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}}\right\} = 1,$$

and therefore

$$\pi = 2\lim_{n\to\infty}\left\{\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right\}.$$

Solution to Question 9. — Vertical cross sections of the smaller part are rectangles of width $2\sqrt{r^2-x^2}$ and height hx/r, for $0 \leqslant x \leqslant r$, where x is the distance between the diameter of the base and the segment in which the cross section meets the base. Therefore, the volume of the smaller part is equal to $2h/r$ times

$$\int_0^r x\sqrt{r^2-x^2}\,dx = -\frac{1}{3}(r^2-x^2)^{3/2}\Big|_0^r = \frac{1}{3}r^3,$$

or $\frac{2}{3}r^2h$ (where the integral is computed by inspection).