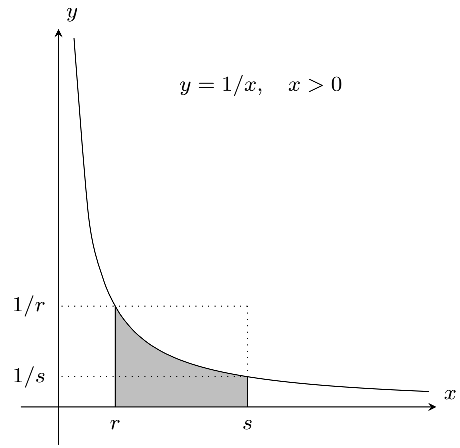
LOGARITHMIC AND EXPONENTIAL FUNCTIONS

1. **Area under the hyperbola.** For positive real numbers r and s with $r < s$, let $A(r,s)$ denote the area of the region $\{\,(x,y)\colon r \leqslant x \leqslant s \text{ and } 0 \leqslant y \leqslant 1/x\,\}$; *i.e.*, the area of the region below the hyperbola $y = 1/x$, $x > 0$, and above the interval $[\,r,s\,]$ on the x-axis. A typical region is shaded in the figure below.



The shaded region includes a rectangle of height $1/s$ and base $s-r$, and is included in a rectangle of height $1/r$ and base $s-r$. It follows that

$$\frac{s-r}{s} < A(r,s) < \frac{s-r}{r}. \qquad (*)$$

Dividing $[\,r,s\,]$ into n subintervals of equal length, applying the estimate $(*)$ to the region over each subinterval, and adding the results, gives an estimate

$$L_n < A(r,s) < U_n,$$

where

$$0 < U_n - L_n < \frac{(s-r)^2}{r^2 n}.$$

This implies that $A(r,s)$ is well-defined, and gives a means of approximating its value to any degree of accuracy.

Multiplying r and s by $t > 0$ leaves the basic estimate $(*)$, and hence L_n and U_n, unchanged; therefore

$$A(r,s) = A(tr,ts).$$

This identity implies that

$$A(1/x,1) = A(1,x), \qquad A(1,xy) = A(1,x) + A(1,y) \qquad \text{and} \qquad A(1,x^\alpha) = \alpha A(1,x), \qquad (1.1)$$

where $x, y > 1$ are real numbers, and α is a positive rational number. In particular, since $A(1,2) > \frac{1}{2}$, it follows that $A(2^{-n},1) = A(1,2^n) = nA(1,2) > \frac{1}{2}n$, and therefore

$$\lim_{x\to 0^+} A(x,1) = \infty \qquad \text{and} \qquad \lim_{x\to\infty} A(1,x) = \infty. \qquad (1.2)$$

The basic estimate $(*)$ implies that, for $0 < r < s$,

$$\frac{1}{s} < \frac{A(r,s)}{s-r} < \frac{1}{r}, \qquad \text{and so} \qquad \lim_{r\to s^-} \frac{A(r,s)}{s-r} = \frac{1}{s} \qquad \text{and} \qquad \lim_{s\to r^+} \frac{A(r,s)}{s-r} = \frac{1}{r}. \qquad (1.3)$$

2. **The logarithm.** The *logarithm* (or *natural logarithm*) function is denoted by log or ln, and is defined by

$$\log x = \ln x = \begin{cases} A(1,x) & \text{if } x \geqslant 1, \text{ and} \\ -A(x,1) & \text{if } 0 < x < 1. \end{cases}$$

In other words, $\log x$ is the area under the hyperbola over $[\,1,x\,]$ if $x \geqslant 1$, and minus the area under the hyperbola over $[\,x,1\,]$ if $0 < x < 1$. The logarithm is a strictly increasing function, since if $0 < r < s$ then $\log s - \log r = A(r,s) > 0$, and so $\log r < \log s$. From $A(1,1) = 0$, and the identities (1.1), follow

$$\begin{aligned} \log 1 &= 0, & \log(1/x) &= -\log x, \\ \log(xy) &= \log x + \log y \quad \text{and} & \log(x^\alpha) &= \alpha \log x, \end{aligned} \qquad (2.1)$$

where x and y are positive real numbers, and α is a rational number. The second and third of these identities imply that $\log(x/y) = \log x - \log y$. From the limits (1.2) follow

$$\lim_{x\to 0^+} \log x = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \log x = \infty. \qquad (2.2)$$

The limits (1.3) imply, using the definition of derivative, that

$$\frac{d}{dx}\{\log x\} = \frac{1}{x}, \qquad \text{and in particular} \qquad \lim_{t\to 1} \frac{\log t}{t-1} = 1. \qquad (2.3)$$

Since logarithm function is differentiable, and therefore continuous, on its domain $(\,0,\infty\,)$, the Intermediate Value Theorem and the limits (2.2) imply that the range of the logarithm function is $\mathbb{R}$.

3. **The number e.** The logarithm function is strictly increasing and its range is $\mathbb{R}$, so there is a unique real number, called e, such that $\log e = 1$. The exact value of e is given by many limit formulæ. Two are

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \qquad \text{and} \qquad e = \lim_{n\to\infty} \left\{1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots + \frac{1}{1\cdot 2\cdot 3\cdots n}\right\}.$$

The first limit formula is obtained by combining (2.3), with $t = 1 + 1/n$, and the last identity of (2.1). The second limit formula is obtained by expanding the expression in the first limit and estimating the result. Like π, e is a transcendental number, meaning that it is not a solution of any polynomial equation with integer coefficients. (This was discovered near the end of the nineteenth century.) In particular, e is not a rational number (something which is easily deduced from the second limit formula). The value of e is approximately $2.7182818284590452353602874713526624977572470936999595749669676277240766303535476$.

4. **The exponential function.** The logarithm function is strictly increasing, so it has an inverse, which is called the *exponential* function and is denoted by exp. The exponential function is defined by

$$y = \exp x \quad \text{if, and only if} \quad \log y = x, \qquad \text{or equivalently,} \quad \exp(\log y) = y \quad \text{and} \quad \log(\exp x) = x,$$

for all real numbers x and all positive real numbers y. Its domain is $\mathbb{R}$ and its range is $(\,0,\infty\,)$. In terms of the exponential function, the identities (2.1) and the definition of e become

$$\begin{aligned} &\exp 0 = 1, \qquad \exp 1 = e, & \exp(-x) &= \frac{1}{\exp x}, \\ &\exp(x+y) = (\exp x)(\exp y) \quad \text{and} & \exp(\alpha x) &= (\exp x)^\alpha, \end{aligned} \qquad (4.1)$$

where x and y are real numbers and α is a rational number. The second and third of these identities imply that $\exp(x-y) = (\exp x)/(\exp y)$. Moreover, if one *defines*

$$x^\alpha = \exp(\alpha \log x)$$

for $x > 0$, then the last identities of (2.1) and (4.1) remain valid for all real values of α and, since $\log e = 1$, one has $\exp x = e^x$ for all real numbers x. Using this notation, the defining properties of the exponential function become

$$y = e^x \quad \text{if, and only if,} \quad \log y = x, \quad \text{or equivalently,} \quad e^{\log y} = y \quad \text{and} \quad \log(e^x) = x,$$

for all real numbers x and all positive real numbers y, and the characteristic identities (4.1) become

$$e^0 = 1, \quad e^1 = e, \quad e^{x+y} = e^x e^y, \quad e^{x-y} = e^x/e^y, \quad e^{-x} = 1/e^x \quad \text{and} \quad e^{\alpha x} = (e^x)^\alpha,$$

for all real numbers x, y and α. Expressed in terms of the exponential function, the limits (2.2) become

$$\lim_{x\to-\infty} \exp x = \lim_{x\to-\infty} e^x = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \exp x = \lim_{x\to\infty} e^x = \infty. \qquad (4.2)$$

If $y = \exp x$ then $\log y = x$, and so $1/y = dx/dy$ by (2.3). Therefore, $y = dy/dx$, *i.e.,*

$$\frac{d}{dx}\{\exp x\} = \exp x, \ \text{or} \ \frac{d}{dx}\{e^x\} = e^x; \quad \text{in particular,} \quad \lim_{t\to 0} \frac{\exp t - 1}{t} = \lim_{t\to 0} \frac{e^t - 1}{t} = 1. \quad (4.3)$$

5. **Logarithmic differentiation** refers to the technique of using the formula

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\{\log|y|\}, \qquad \text{or equivalently} \qquad \frac{dy}{dx} = y\frac{d}{dx}\{\log|y|\},$$

for $y \neq 0$, to calculate a derivative. This technique simplifies certain calculations (*e.g.*, if y involves multiple products). The formula is a consequence of the Chain Rule and the fact that $\frac{d}{dy}\{|y|\} = |y|/y$ if $y \neq 0$.

6. **Other bases.** For $a > 0$, $a \neq 1$, the *logarithmic and exponential functions with base a* are defined by

$$\log_a(x) = \frac{\log x}{\log a} \qquad \text{and} \qquad \exp_a(x) = a^x = e^{x\log a}.$$

These functions are inverses, $\log_a$ satisfies (2.1) and $\log_a(a) = 1$; $\exp_a$ satisfies (4.1) with one difference, namely $\exp_a(1) = a$.

7. **Notation.** Some calculators and children's books use log to denote $\log_{10}$. Grown-ups, as well as systems designed by and for them, understand that ln and log refer to the natural logarithm. Our textbook writes ln for the natural logarithm and does not write log without a subscript. In class we will usually write ln, but if we do write log without a subscript, it will refer to the natural logarithm.