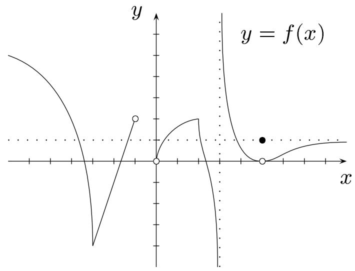
CEGEP JOHN ABBOTT COLLEGE

CALCULUS I (SCIENCE)

(MATHEMATICS 201-NYA/∞)

Autumn 2002

1. Refer to the accompanying graph of $y = f(x)$ in order to answer the following questions (each mark on an axis represents one unit).



   (a) What is the domain of f?
   (b) What is the range of f?
   (c) Evaluate, if possible,
      (i) $\lim_{x\to 3^+} f(x)$
      (ii) $\lim_{x\to\infty} f(x)$
      (iii) $\lim_{x\to -1} f(x)$
      (iv) $\lim_{x\to 5} f(x)$
      (v) $f(5)$
   (d) At which values of x, if any, is f discontinuous?
   (e) At which values of x, if any, is f continuous but not differentiable?

2. Evaluate each of the following limits. Use the terms ∞, $-\infty$, and *does not exist*, as appropriate.

   (a) $\displaystyle\lim_{x\to -2} \frac{3x^2+x-10}{2x^2+x-6}$ (b) $\displaystyle\lim_{x\to -\infty} \frac{x+2}{\sqrt{3x^2+1}}$

   (c) $\displaystyle\lim_{x\to -2^-} \frac{x^2-5}{x^2-4}$ (d) $\displaystyle\lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3}}{h}$

3. Given
$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x^2+2 & \text{if } -1 \le x < 3 \\ e^x+2 & \text{if } x \ge 3 \end{cases}$$
Determine all points of discontinuity of the function $f(x)$. Justify your answer.

4. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{1}{2-3x}$.

5. Find $\dfrac{dy}{dx}$ for each of the following:

   (a) $y = \dfrac{3x}{x^2+1}$ (b) $y = x^2\tan x + e^{\sec 2x}$

   (c) $y = \ln\left(\dfrac{e^{x^2}\sqrt{3+2x}}{2+7x^2}\right)$ (d) $y = (\sin x)^{\cos x}$

   (e) $y = \left(\dfrac{x+1}{x+2}\right)(2x-5)$ (f) $x^3 - 2x^2y + y^4 = 1$

6. For which value(s) of x does the graph of $y = (3x+5)^4(4-x)^3$ have a horizontal tangent line?

7. A particle moves along a straight line with equation of motion $s = t^2 \ln t$. Find the acceleration of the particle when the velocity is zero.

8. Find the equation of the line tangent to the graph of $f(x) = \sqrt{x^2+3}$ at the point where $x = 1$.

9. Find all point(s) of inflection of the function $f(x) = x^2e^x$.

10. Find the largest and smallest values of $f(x) = x\sqrt{1-x^2}$ on the interval $\left[\frac{1}{2}, 1\right]$.

11. Find y given $y'' = -1 + \sin t$, $y'(0) = 3$ and $y(2\pi) = 0$.

12. A wire 17 metres long is cut into 2 pieces. One piece is bent to form a square and the other is bent to form a rectangle that is twice as long as it is wide. How should the wire be cut so that the sum of the two areas is minimum?

13. A stone is tossed into a still pond. A circular wave spreads at the rate of 10 m/s. How fast is the area of the pond enclosed by the wave increasing when the edge of the waveform is 20 metres from the center of the point of impact?

14. Given
$$f(x) = \frac{3x}{(1-x)^2},\quad f'(x) = \frac{3+3x}{(1-x)^3},\quad f''(x) = \frac{12+6x}{(1-x)^4}.$$
Graph the function $f(x)$, identifying all intercepts, asymptotes, local extrema and inflection points. Specify intervals where the function is increasing, decreasing, concave up and concave down. *Show all your work.*

15. Evaluate the area between $y = |x-2|$ and the x-axis from $x = 0$ to $x = 3$.

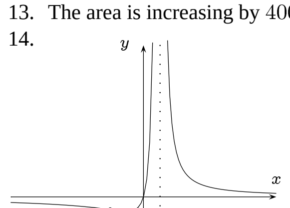
16. Evaluate the following integrals:

   (a) $\displaystyle\int \left(x^{3/5} + \frac{1}{x^3} - \frac{1}{x} + \frac{1}{e^{-x}} + 2\pi\right) dx$ (b) $\displaystyle\int_1^3 \frac{t^3-t}{t^2}\, dt$

   (c) $\displaystyle\int \sec x(\sec x - \tan x)\, dx$

17. Use differentiation to verify that:
$$\int \frac{1}{(4-x^2)^{3/2}}\, dx = \frac{x}{4\sqrt{4-x^2}} + C$$

ANSWERS

1. (a) $\mathbb{R}\setminus([-1,0]\cup\{3\})$, (b) $\mathbb{R}$, (c) (i) ∞, (ii) 1, (iii) undefined, (iv) 0, (v) 1, (d) All $x \in [-1,0]\cup\{3,\ 5\}$, (e) $-3,\ 2$.
2. (a) $11/7$, (b) $-1/\sqrt{3}$, (c) $-\infty$, (d) $1/(2\sqrt{3})$.
3. f has a jump discontinuity at $x = 3$.
4. $f'(x) = \displaystyle\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{2-3(x+h)} - \frac{1}{2-3x}\right) = \cdots = \frac{3}{(2-3x)^2}$
5. (a) $\dfrac{3(1-x^2)}{(x^2+1)^2}$, (b) $2x\tan x + x^2\sec^2 x + 2\sec 2x\tan 2x\, e^{\sec 2x}$, (c) $2x + \dfrac{1}{3+2x} - \dfrac{14x}{2+7x^2}$, (d) $(\sin x)^{\cos x}\left\{\dfrac{\cos^2 x}{\sin x} - \sin x\ln\sin x\right\}$, (e) $\dfrac{2x^2+8x-1}{(x+2)^2}$, (f) $\dfrac{x(3x-4y)}{2(x^2-2y^3)}$.
6. At $x = -5/3,\ 7/11,\ 4$.
7. $a = 2$
8. $x - 2y + 3 = 0$
9. $\left(-2\pm\sqrt{2}, (6\pm 4\sqrt{2})\, e^{-2\pm\sqrt{2}}\right)$
10. The largest value is $\frac{1}{2}$ and the smallest value is 0.
11. $y = -t^2/2 - \sin t + 4t + 2\pi(\pi - 4)$
12. Cut the wire so as to use 8 m for the square and 9 m for the rectangle.
13. The area is increasing by $400\ \mathrm{m}^2/\mathrm{s}$.
14. 

The only intercept is $(0,0)$, the asymptotes are $x = 1$ (vertical) and $y = 0$ (horizontal), the only local extremum is $\left(-1, -\frac{3}{4}\right)$, and the only inflection point is $\left(-2, -\frac{2}{3}\right)$. f is decreasing on $(-\infty,-1)$ and on $(1,\infty)$, and increasing on $(-1,1)$. f is concave down on $(-\infty,-2)$, and concave up on $(-2,1)$ and $(1,\infty)$.
15. The area is $5/2$ square units.
16. (a) $\frac{5}{8}x^{8/5} - 1/(2x^2) - \ln|x| + e^x + 2\pi x + C$, (b) $4 - \ln 3$, (c) $\tan x - \sec x + C$.
17. $\dfrac{d}{dx}\left\{\dfrac{x}{4\sqrt{4-x^2}}\right\} = \dfrac{1}{4}\left\{\dfrac{1}{\sqrt{4-x^2}} + \dfrac{x^2}{(4-x^2)^{3/2}}\right\}$
$= \frac{1}{4}\cdot\dfrac{4-x^2+x^2}{(4-x^2)^{3/2}} = \dfrac{1}{(4-x^2)^{3/2}}$.