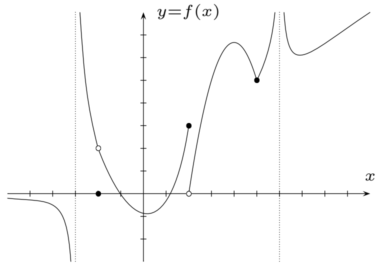
CALCULUS I (SCIENCE)

(MATHEMATICS 201–NYA/∞)

1. Refer to the given graph of $y = f(x)$ to answer the following questions. (Each mark on an axis represents one unit.)
   (a) Evaluate, if possible, using the terms ∞, $-\infty$ and "does not exist" where appropriate.
   (i) $\lim_{x\to -3^-} f(x)$
   (ii) $\lim_{x\to -2} f(x)$
   (iii) $\lim_{x\to 2} f(x)$
   (iv) $\lim_{x\to 6} f(x)$
   (v) $\lim_{x\to -\infty} f(x)$



   (b) For which values of x, if any, is f discontinuous?
   (c) For which values of x, if any, is f continuous but not differentiable?

2. Evaluate each of the following limits. Use the terms ∞, $-\infty$ and "does not exist" as appropriate.
   (a) $\lim_{x\to -3} \dfrac{2x^2+5x-3}{x^2+2x-3}$ (b) $\lim_{x\to -1} \dfrac{3x+1}{x^2-x+1}$
   (c) $\lim_{x\to 0} \dfrac{5-\sqrt{x+25}}{2x}$ (d) $\lim_{x\to -\infty} \dfrac{\sqrt{4x^2+9}}{2x+1}$

3. State whether the following statements are **true** or **false**. Justify.
   (a) The graph of a function will never cross a horizontal asymptote.
   (b) If $f(x)$ is continuous at $x = a$, then f is differentiable at $x = a$.
   (c) For a continuous function f, if $f(a) > 0$ and $f(b) < 0$ then $f(c) = 0$ for some $c \in (a, b)$.
   (d) It is possible to sketch the graph of a continuous function $f(x)$ such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) < 0$ for all $x \in \mathbb{R}$.

4. (a) Define the phrase "f is continuous at $x = a$."
   (b) Using this definition, determine whether f is continous at $x = 5$, where

$$f(x) = \begin{cases} (x-5)^2 & \text{if } x < 5, \\ \ln(x-4) & \text{if } x \geqslant 5. \end{cases}$$

5. Give, with justification, the equations of all asymptotes of

$$f(x) = \frac{(2x-1)(x+3)}{x^2-x-6}.$$

6. Use the definition of the derivative to find $f'(x)$, where $f(x) = \sqrt{3x-5}$.

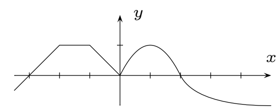
7. Find $\frac{dy}{dx}$ for each of the following. *Do not simplify your answers*.
   (a) $y = x^5 - \dfrac{1}{3x^2} + 3^x + \dfrac{2}{\sqrt[3]{x}} + e$ (b) $y = \cot(5x^2 - e^x)$
   (c) $y = (2x+1)^x$ (d) $y = \ln\left\{\dfrac{(x^3+1)^2}{(3x+5)(x^2+4)^3}\right\}$
   (e) $y = \left(\dfrac{5x^2}{9x^3+2}\right)^3$ (f) $x^3 - 2x^2y + 3xy^2 = 42$

8. Find the equation of the tangent line to the graph of $y = x + e^{5x}$ at $(0, 1)$.

9. Find the second derivative of $f(x) = \dfrac{3x}{x-1}$.

10. The position of an object is given by the function $x = 36t - \frac{1}{3}t^3$, for $0 \leqslant t \leqslant 10$, where x is measured in metres and t is measured in seconds. (a) Find the acceleration when the velocity is zero. (b) What does negative acceleration mean?

11. Sketch the graph of the derivative of the function whose graph is given below.



12. An airplane flys at a constant altitude of 12 km along a straight path that will take it directly over a radar station. When the distance between the plane and the radar station is 13 km, that distance is decreasing at a rate of 300 km/hr. What is the ground speed of the plane at this instant?

13. Let $f(x) = x\sqrt{32-x^2}$. (a) Find all critical numbers of f. (b) Find the absolute maximum and absolute minimum values of f on the interval $[-5, 0]$.

14. Given $f(x)$ and its derivatives, do the following. Graph the function f, identifying all intercepts, local extrema, and points of inflection. Specify intervals where the function is increasing, decreasing, concave up and concave down.

$$f(x) = (x-4)\sqrt[3]{x}, \quad f'(x) = \frac{4(x-1)}{3\sqrt[3]{x^2}}, \quad f''(x) = \frac{4(x+2)}{9\sqrt[3]{x^5}}.$$

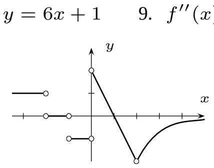
15. A closed rectangular container with a square base is to have a volume of $2250\,\text{cm}^3$. The material for the top and bottom of the container will cost \$2 per cm^2 and the material for the sides will cost \$3 per cm^2. Find the dimensions of the container of least cost.

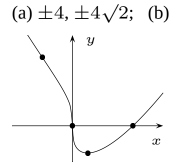
16. Given $f''(t) = \cos t - \sin t$, find $f(t)$ if $f'(0) = 5$ and $f(0) = 0$.

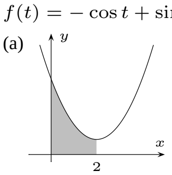
17. (a) Draw a sketch of the region bounded by $y = x^2 - 4x + 5$ and the x-axis, between $x = 0$ and $x = 2$. (b) Calculate the exact area of the region you sketched in part (a).

18. Evaluate the following integrals. (a) $\displaystyle\int \left(2e^x + \frac{3}{x} - \frac{1}{x^2} + \frac{1}{2\sqrt{x}}\right) dx$
    (b) $\displaystyle\int \cos\vartheta(1 + \sec\vartheta - \tan\vartheta)\, d\vartheta$ (c) $\displaystyle\int_1^3 x(1+2x)\, dx$

ANSWERS

1. (a) (i) $-\infty$, (ii) 2, (iii) DNE, (iv) ∞, (v) 0; (b) $-3, \pm 2, 6$; (c) 5.
2. (a) $\frac{7}{4}$; (b) $-\frac{2}{3}$; (c) $-\frac{1}{20}$; (d) -1.
3. (a) **false**; (b) **false**; (c) **true**; (d) **false**.
4. (a) $\lim_{x\to a} f(x) = f(a)$; (b) f is continuous at 5.
5. VA: $x = -2$, $x = 3$; HA: $y = 2$.
6. $f'(x) = \lim_{h\to 0} \left\{\left(\sqrt{3(x+h)-5} - \sqrt{3x-5}\right)/h\right\} = \cdots = \dfrac{3}{2\sqrt{3x-5}}$
7. (a) $5x^4 + \frac{2}{3}x^{-3} + 3^x \ln 3 - \frac{2}{3}x^{-4/3}$; (b) $(e^x - 10x)\csc^2(5x^2 - e^x)$;
   (c) $(2x+1)^{x-1}\{2x + (2x+1)\ln(2x+1)\}$; (d) $\dfrac{6x^2}{x^3+1} - \dfrac{3}{3x+5} - \dfrac{6x}{x^2+4}$;
   (e) $3\left(\dfrac{5x^2}{9x^3+2}\right)^2 \dfrac{10x(9x^3+2)-(5x^2)(27x^2)}{(9x^3+2)^2}$; (f) $\dfrac{4xy-3x^2-3y^2}{6xy-2x^2}$.
8. $y = 6x + 1$
9. $f''(x) = 6(x-1)^{-3}$
10. (a) $a = -12\ \text{m/s}^2$; (b) The velocity is decreasing.
11. 

12. The plane is travelling at $780\ \text{km/hr}$ when it is 13 km from the radar station.
13. (a) $\pm 4, \pm 4\sqrt{2}$; (b) MAX: $f(0) = 0$; MIN: $f(-4) = -16$.
14. 

    Intercepts: $(0, 0)$; $(4, 0)$.
    Extrema: $(1, -3)$ (local min). There are no global extrema.
    IP's: $(-2, 6\sqrt[3]{2})$; $(0, 0)$.
15. To minimize the cost, the dimensions of the box should be 15 cm by 15 cm by 10 cm.
16. $f(t) = -\cos t + \sin t + 4t + 1$
17. (a) 

    (b) $14/3$ square units
18. (a) $2e^x + 3\ln|x| + 1/x + \sqrt{x} + C$; (b) $\sin\vartheta + \vartheta + \cos\vartheta + C$; (c) $64/3$.