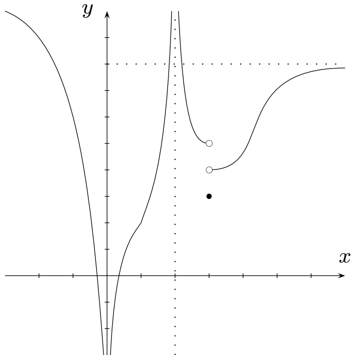
1. Refer to the sketch below (with unit lengths marked along the coordinate axes) to evaluate the following. If a value does not exist, state in which way (∞, $-\infty$ or "does not exist").



a. $\lim_{x\to-\infty} f(x)$
b. $\lim_{x\to 0} f(x)$
c. $\lim_{x\to 2} f(x)$
d. $\lim_{x\to 3^-} f(x)$
e. $\lim_{x\to 3^+} f(x)$
f. $f(3)$
g. $\lim_{x\to\infty} f(x)$
h. List the values of x where f is discontinuous.

2. Evaluate the limits. Use the symbols ∞ or $-\infty$ where appropriate.

a. $\lim_{x\to\infty} \dfrac{6-11x+3x^2}{3+14x-5x^2}$ b. $\lim_{x\to 3} \dfrac{6-11x+3x^2}{3+14x-5x^2}$

c. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$ d. $\lim_{x\to 0} \dfrac{\sin 2x}{\tan x}$ e. $\lim_{x\to 0+} \dfrac{3|x|-x}{x^2}$

3. Find all asymptotes of the graph of f, where

$$f(x) = \frac{3x+\sqrt{x^2+1}}{2x-1}.$$

4. a. Define "f is continuous at a," where f is a function and a is a real number.

b. Determine whether f is continuous at 0. Justify your answer completely.

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x<0, \\ 2 & \text{if } x=0, \text{ and} \\ \cos x & \text{if } x>0. \end{cases}$$

c. Sketch the graph of the function f from part b.

5. Sketch the graph of a function f that is continuous but not differentiable at 0.

6. State the (limit) definition of the derivative. Use the definition of the derivative to find the derivative of f, where

$$f(x) = \frac{2}{1-x}.$$

7. For each of the following functions calculate the derivative, $\dfrac{dy}{dx}$.

a. $y = \dfrac{x^7}{7} - \dfrac{2}{\sqrt[5]{x^2}} + \dfrac{\sqrt{x}}{3} - \ln 2 + 3^x$ b. $y = x^8 \sec x$

c. $y = \dfrac{\tan^3(x+1)}{\ln(3x-2)}$ d. $y = \sqrt{\sin(e^x+e^2)}$ e. $y = (x+1)^{(x^2+1)}$

8. Find an equation of the line tangent to the graph of

$$y = \frac{\cos(x-1)}{x+1}.$$

at the point where $x = 1$.

9. Given $x^3 + y^3 = 6xy + 1$:

a. find $\dfrac{dy}{dx}$ using implicit differentiation;

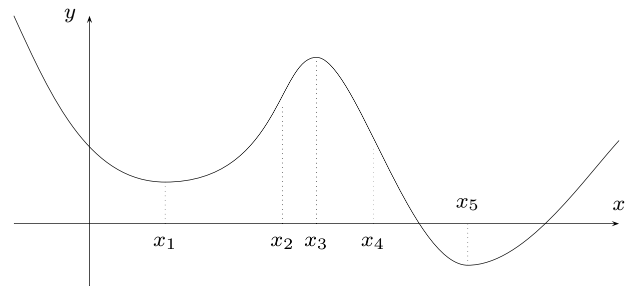
b. find an equation of the tangent line to the curve at each x-intercept;

c. find all points on the curve at which the tangent line is horizontal.

10. Find the global extrema of h on the interval $[-1, 1]$, where

$$h(x) = \frac{x^2-1}{x^2+1}.$$

11. The graph of f is given below.



a. At which values of x does f'
i. change sign? ii. have a local maximum? iii. have a local minimum?

b. Sketch a rough graph of f'.

c. At which values of x does f'' change sign?

12. A boat is pulled into a dock by a rope attached to the bow of the boat, passing through a pulley on the dock that is 10 metres higher than the bow of the boat. The rope is pulled in at a constant rate of 5 metres per minute. At what rate is the angle between the water and the rope changing at the instant when the length of rope between the pulley and the bow of the boat is 26 metres?

13. Sketch the graph of

$$f(x) = \frac{x-2}{x^3}.$$

Identify all intercepts, asymptotes, local extrema and points of inflection. Specify the intervals of monotonicity and concavity, and show all relevant work.

14. A rectangular box has an open top, a square base, and a surface area of 147 square metres; find the dimensions of the box if it is to have the maximum possible volume.

15. A number p in the domain of a function f is called a *fixed point* of f if $f(p) = p$. Suppose that the function f is differentiable on $\mathbb{R}$, and satisfies

$$f(0) = 1, \quad f(1) = 0 \quad \text{and} \quad f'(x) < \tfrac{2}{3} \quad \text{for all real numbers } x.$$

a. Show that f has a fixed point. (*Hint*: Use the Intermediate Value Theorem.)

b. Show that f has exactly one fixed point. (*Hint*: Use Rolle's Theorem or the Mean Value Theorem.)

16. Given that $f'(x) = x + \sin x$, and $f(0) = 3$, find $f(x)$.

17. Evaluate the following integrals.

a. $\displaystyle\int \frac{x^4-4x-1}{2x^2}\,dx$ b. $\displaystyle\int \left(\frac{4}{t^5} - \frac{t^5}{4} + 5e^{-t} - \frac{1}{e^5}\right) dt$

c. $\displaystyle\int \sec x(\tan x - \sec x)\,dx$ d. $\displaystyle\int_1^4 (\sqrt{x}+2)^2\,dx$

18. a. Evaluate the Riemann sum for $f(x) = 2^x$ on $[-1, 3]$, with four subintervals of equal length, taking sample points to be right endpoints.

b. Evaluate

$$\int_{-1}^{3} 2^x\,dx$$

as a limit of (left endpoint) Riemann sums (*i.e.*, do not use the Fundamental Theorem of Calculus).

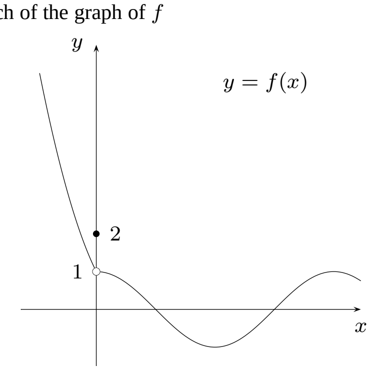
19. Find the area of the region bounded by the curve $y = x^2 - 2x$ and the x-axis.

1. By inspecting the given sketch: a. $\lim_{x\to-\infty} f(x) = \infty$; b. $\lim_{x\to 0} f(x) = -\infty$; c. $\lim_{x\to 2} f(x) = \infty$; d. $\lim_{x\to 3^-} f(x) = 5$; e. $\lim_{x\to 3^+} f(x) = 4$; f. $f(3) = 3$; g. $\lim_{x\to\infty} f(x) = 8$; h. f is discontinuous at 0, 2 and 3.

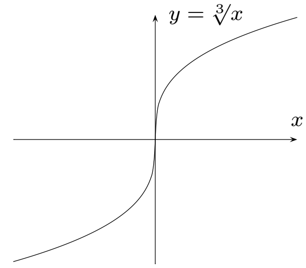
2. a. Extracting dominant terms gives
$$\lim_{x\to\infty} \frac{6-11x+3x^2}{3+14x-5x^2} = \lim_{x\to\infty} \frac{6/x^2-11/x+3}{3/x^2+14/x-5} = -\tfrac{3}{5},$$
since $x^{-p} \to 0$ as $x \to \infty$ for any positive real number p.
b. Factoring and simplifying gives
$$\lim_{x\to 3} \frac{6-11x+3x^2}{3+14x-5x^2} = \lim_{x\to 3} \frac{(3-x)(2-3x)}{(3-x)(1+5x)} = \lim_{x\to 3} \frac{2-3x}{1+5x} = -\tfrac{7}{16}.$$
c. Rationalizing the numerator of the expression in the limit and simplifying gives
$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \tfrac{1}{6}.$$
d. Applying the double angle identity for the sine function and simplifying gives
$$\lim_{x\to 0} \frac{\sin 2x}{\tan x} = \lim_{x\to 0} \left\{ 2\sin x \cos x \cdot \frac{\cos x}{\sin x} \right\} = 2 \lim_{x\to 0} \cos^2 x = 2.$$
e. Since $|x| = x$ if $x > 0$, one has
$$\lim_{x\to 0^+} \frac{3|x|-x}{x^2} = \lim_{x\to 0^+} \frac{2x}{x^2} = \lim_{x\to 0^+} \frac{2}{x} = \infty.$$

3. If $f(x) = \dfrac{3x+\sqrt{x^2+1}}{2x-1}$, then
$$\lim_{x\to-\infty} \frac{3x+\sqrt{x^2+1}}{2x-1} = \lim_{x\to-\infty} \frac{3-\sqrt{1+1/x^2}}{2-1/x} = 1,$$
since $|x| = -x$ if $x < 0$, and
$$\lim_{x\to\infty} \frac{3x+\sqrt{x^2+1}}{2x-1} = \lim_{x\to\infty} \frac{3+\sqrt{1+1/x^2}}{2-1/x} = 2,$$
since $|x| = x$ if $x > 0$. Therefore, $y = 1$ and $y = 2$ are the equations of the horizontal asymptotes of the graph of f. Next, observe that, since $3x+\sqrt{x^2+1} \to \frac{1}{2}(3+\sqrt{5})$ and $2x-1 \to 0^{\pm}$ as $x \to \frac{1}{2}^{\pm}$,
$$\lim_{x\to\frac{1}{2}^{\pm}} \frac{3x+\sqrt{x^2+1}}{2x-1} = \pm\infty,$$
and so f has an infinite discontinuity at $\frac{1}{2}$. Since f is otherwise continuous, the graph of f has one vertical asymptote, with equation $x = \frac{1}{2}$.

4. a. Let f be a function and let a be a real number in the domain of f; f is continuous at a if the limit as $x \to a$ of $f(x)$ is equal to $f(a)$.
b. Since
$$\lim_{x\to 0^-} f(x) = (0-1)^2 = 1, \quad f(0) = 2 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \cos 0 = 1,$$
it follows that f has a removable discontinuity at 0 (and so f is not continuous at 0).
c. Below is a sketch of the graph of f



5. The cube root function, whose graph is below, is an example of such a function.



6. The derivative of a function f is the function f' defined by
$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x};$$
the domain of f' is its natural domain, *i.e.*, the set of all real numbers x such that the foregoing limit is defined.

7. a. Differentiating term by term gives
$$\frac{dy}{dx} = x^6 + \tfrac{4}{5}\sqrt[5]{x^{-7}} + \tfrac{1}{6}\sqrt{x^{-1}} + 3^x \log 3.$$
b. By the Product Rule,
$$\frac{dy}{dx} = 8x^7 \sec x + x^8 \sec x \tan x = x^7(8 + x\tan x)\sec x.$$
c. Writing $y = \tan^3(x+1)\{\ln(3x-2)\}^{-1}$ and applying the Product Rule and the Chain Rule, gives
$$\begin{aligned}\frac{dy}{dx} &= \frac{3\tan^2(x+1)\sec^2(x+1)}{\ln(3x-2)} - \frac{3\tan^3(x+1)}{(3x-2)\{\ln(3x-2)\}^2} \\ &= \frac{3\big((3x-2)\ln(3x-2)\sec^2(x+1) - \tan(x+1)\big)\tan^2(x+1)}{(3x-2)\{\ln(3x-2)\}^2}.\end{aligned}$$
d. By the Chain Rule,
$$\frac{dy}{dx} = \frac{e^x\cos(e^x+e^2)}{2\sqrt{\sin(e^x+e^2)}}.$$
e. Writing $y = (x+1)^{x^2+1} = e^{(x^2+1)\log(x+1)}$, and applying the Chain Rule and the Product Rule, gives
$$\begin{aligned}\frac{dy}{dx} &= e^{(x^2+1)\log(x+1)}\big(2x\log(x+1) + (x^2+1)/(x+1)\big) \\ &= (x+1)^{x^2}\big(1 + x^2 + 2x(x+1)\log(x+1)\big).\end{aligned}$$

8. Where $x = 1$, $y = \frac{1}{2}$, and
$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{-(x+1)\sin(x-1) - \cos(x-1)}{(x+1)^2}\right|_{x=1} = -\tfrac{1}{4};$$
therefore, an equation of the tangent line in question is $x + 4y = 3$.

9. a. Differentiating the given equation with respect to x gives
$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}, \qquad \text{or} \qquad 3(y^2-2x)\frac{dy}{dx} = 3(2y-x^2);$$
therefore,
$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}.$$
b. If $y = 0$ on the given curve, then $x^3 = 1$, so the curve has one x-intercept, namely $(1,0)$. At this point $dy/dx = \frac{1}{2}$, so an equation of the tangent line to the curve at its x-intercept is $x - 2y = 1$.
c. The tangent line is horizontal where $dy/dx = 0$; *i.e.*, where $2y - x^2 = 0$, or $y = \frac{1}{2}x^2$, and $y^2 - 2x \neq 0$. Replacing y by $\frac{1}{2}x^2$ in the equation of the curve yields $x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right) + 1$, *i.e.*, $x^6 - 16x^3 = 8$, or $(x^3-8)^2 = 72$. Hence, $x^3 - 8 = \pm\sqrt{72} = 6\sqrt{2}$, or $x = \sqrt[3]{(8 \pm 6\sqrt{2})}$, and $y = \frac{1}{2}\sqrt[3]{(8 \pm 6\sqrt{2})^2} = \sqrt[3]{(34 \pm 24\sqrt{2})}$. Therefore, the tangent line to the graph of $x^3 + y^3 = 6xy + 1$ is horizontal at the points $\big(\sqrt[3]{(8+6\sqrt{2})}, \sqrt[3]{(34+24\sqrt{2})}\big)$ and $\big(\sqrt[3]{(8-6\sqrt{2})}, \sqrt[3]{(34-24\sqrt{2})}\big)$.

10. If

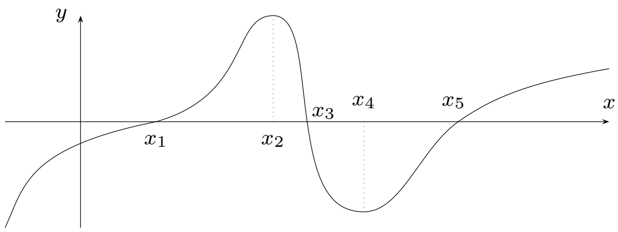
$$h(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}, \qquad \text{then} \qquad h'(x) = \frac{4x}{(x^2+1)^2}$$

by the Reciprocal Rule and linearity of the derivative. Therefore, h has one critical number, 0, in $(-1,1)$. Comparing $h(\pm 1) = 0$ and $h(0) = 1$ reveals that the global minimum value of h on $[-1,1]$ is $h(0) = -1$ and the global maximum value of h on $[-1,1]$ is $h(\pm 1) = 0$.

11. Observe (at least on visible portion of the graph), that f is decreasing to the left of x_1 and on (x_3, x_5), and is otherwise increasing, and that the graph of f is concave down on (x_2, x_4) and is otherwise concave up.

a. The foregoing observations imply that i. f' changes sign at x_1, x_3 and x_5; ii. f' has a local maximum at x_2; iii. f' has a local minimum at x_4.

b. A rough sketch of the graph of f' is given below.



c. f'' changes sign where the graph of f changes concavity, *i.e.*, at x_2 and at x_4.

12. If ℓ denotes the length (in metres) of rope between the pulley and the boat, and ϑ denotes the radian measure of the angle between the water and the rope, then $\sin\vartheta = 10/\ell$, and so

$$\cos\vartheta \frac{d\vartheta}{dt} = -\frac{10}{\ell^2}\frac{d\ell}{dt} = \frac{50}{\ell^2},$$

since the rope is being pulled in at a constant rate of 5 metres per minute (so $d\ell/dt = -5$). When ℓ is 26, $\sin\vartheta = \frac{10}{26} = \frac{5}{13}$, so $\cos\vartheta = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}$, and hence

$$\frac{d\vartheta}{dt} = \frac{50}{26^2}\cdot\frac{13}{12} = \frac{25}{312}.$$

Therefore, the angle between the water and the rope is increasing at a rate of $\frac{25}{312}$ radians per minute when the length of rope between the pulley and the bow of the boat is 26 metres.

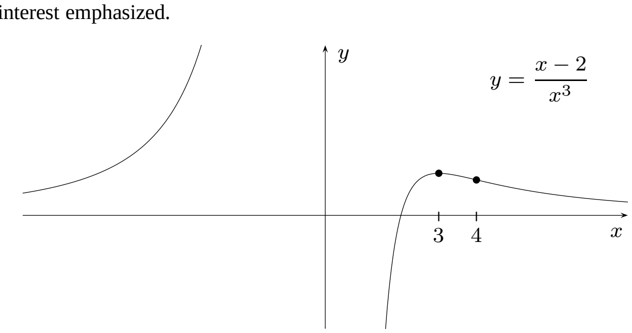
13. The domain of f is $\mathbb{R}\setminus\{0\}$, and $f(x)\to\mp\infty$ as $x\to 0^{\pm}$, so the graph of f has one vertical asymptote with equation $x=0$. Since $f(x) = (1-2/x)/x^2 \to 0$ as $x\to\pm\infty$, the graph of f has one horizontal asymptote, namely the x-axis. The first derivative of f is

$$f'(x) = \frac{d}{dx}\left\{\frac{1}{x^2} - \frac{2}{x^3}\right\} = -\frac{2}{x^3} + \frac{6}{x^4} = \frac{2(3-x)}{x^4},$$

which is positive if $x<3$ and $x\neq 0$, and negative if $x>3$. Therefore, f is increasing on $(-\infty,0)$ and on $(0,3)$, and decreasing on $(3,\infty)$, with a local maximum at $(3, \frac{1}{27})$. The second derivative of f is

$$f''(x) = \frac{d}{dx}\left\{-\frac{2}{x^3} + \frac{6}{x^4}\right\} = \frac{6}{x^4} - \frac{24}{x^5} = \frac{6(x-4)}{x^5},$$

which is negative if $x<4$ and $x\neq 0$, and positive if $x>4$. Therefore, the graph of f is concave down on $(-\infty,0)$ and on $(0,4)$, and concave up on $(4,\infty)$, with a point of inflection at $(4, \frac{1}{32})$. Below is a sketch of the graph of f (not to scale, the y-axis is scaled by a factor of forty relative to the x-axis), with the points of interest emphasized.



14. If x is the length of a side of the base of the box, and y is the height of the box, then its volume is $V = x^2y$, and its surface area is $147 = x^2 + 4xy$. Then $y = (147-x^2)/(4x)$, and so $V = \frac{1}{4}x(147-x^2) = \frac{1}{4}(147x - x^3)$, where $x>0$. Hence,

$$\frac{dV}{dx} = \tfrac{1}{4}(147-3x^2) = \tfrac{3}{4}(49-x^2),$$

whose positive zero is 7. Since

$$\frac{d^2V}{dx^2} = -\tfrac{3}{2}x$$

is negative for $x>0$, V has a global maximum value on $(0,\infty)$ at $x=7$ by the Second Derivative Test. When $x=7$, $y=(147-49)/(28) = \frac{7}{2}$. Therefore, a box as described with the largest possible volume will have a 7 m × 7 m base and height $\frac{7}{2}$ m.

15. a. Let $g(x) = f(x) - x$, since f is differentiable on $\mathbb{R}$, g is continuous on $[0,1]$. Since $g(0) = f(0) - 0 = 1 > 0$ and $g(1) = f(1) - 1 = -1 < 0$, the Intermediate Value Theorem implies that $g(\xi) = 0$ for some real number ξ in $(0,1)$. Hence, $f(\xi) - \xi = 0$, or $f(\xi) = \xi$, so f has a fixed point.

b. Suppose that p and q are fixed points of f, with $p<q$. Since f is differentiable on $\mathbb{R}$, f is continuous on $[p,q]$ and differentiable on (p,q), and so the Mean Value Theorem implies that there is a real number ξ in (p,q) such that $f(q) - f(p) = f'(\xi)(q-p)$, or $q - p = f'(\xi)(q-p)$ since p and q are fixed points of f. This last equation implies that $f'(\xi) = 1$ (since $p\neq q$), which contradicts the fact that $f'(x) < \frac{2}{3}$ for all real numbers x. Therefore f cannot have two different fixed points. Since f has a fixed point by part a, it follows that f has exactly one fixed point.

16. By the Fundamental Theorem of Calculus,

$$f(x) = 3 + \int_0^x (t+\sin t)\,dt = 3 + \left(\tfrac{1}{2}t^2 - \cos t\right)\Big|_0^x = 4 + \tfrac{1}{2}x^2 - \cos x.$$

17. a. Dividing and integrating term by term yields

$$\int\left(\tfrac{1}{2}x^2 - 2x^{-1} - \tfrac{1}{2}x^{-2}\right)dx = \tfrac{1}{6}x^3 - 2\log|x| + \tfrac{1}{2}x^{-1} + C.$$

b. Integrating term by term gives

$$\int\left(4t^{-5} - \tfrac{1}{4}t^5 + 5e^{-t} - e^{-5}\right)dt = -t^{-4} - \tfrac{1}{24}t^6 - 5e^{-t} - e^{-5}t + C.$$

c. Expanding the product and integrating term by term gives

$$\int(\sec x\tan x - \sec^2 x)\,dx = \sec x - \tan x + C.$$

d. Expanding and integrating term by term gives

$$\int_1^4 (x + 4\sqrt{x} + 4)\,dx = \left(\tfrac{1}{2}x^2 + \tfrac{8}{3}x^{3/2} + 4x\right)\Big|_1^4 = \tfrac{229}{6}.$$

18. a. Dividing $[-1,3]$ into four subintervals of equal length gives $\Delta x = 1$, and $x_i = -1+i$, for $i = 0,\ldots,4$, or $-1, 0, 1, 2, 3$. The corresponding right endpoint sum is

$$\mathscr{R}_4 = 1\cdot\left\{f(0) + f(1) + f(2) + f(3)\right\} = 2^0 + 2^1 + 2^2 + 2^3 = 15.$$

b. Dividing $[-1,3]$ into n subintervals of equal length gives $\Delta x = \frac{4}{n}$ and $x_i = -1 + \frac{4}{n}i$, for $i = 0,\ldots,n$. The corresponding left endpoint sum is

$$\mathscr{L}_n = \frac{4}{n}\sum_{i=0}^{n-1} 2^{-1+4i/n} = \frac{2}{n}\sum_{i=0}^{n-1}\left(2^{4/n}\right)^i = \frac{2}{n}\cdot\frac{2^4-1}{2^{4/n}-1},$$

by extracting a common factor and applying the summation formula for an exponential function (*i.e.*, a geometric progression). If $t = (4\log 2)/n$, then $2/n = t/(2\log 2)$ and $2^{4/n} = e^t$; therefore,

$$\lim_{n\to\infty}\mathscr{L}_n = \frac{15}{2\log 2}\lim_{t\to 0^+}\frac{t}{e^t-1} = \frac{15}{2\log 2}, \text{ and so, } \int_{-1}^3 2^x\,dx = \frac{15}{2\log 2}.$$

19. The graph of $y = x^2 - 2x = x(x-2)$ is below the x-axis on $[0,2]$ and otherwise above the x-axis. So the area of the region bounded by this curve and the x-axis is

$$-\int_0^2 (x^2-2x)\,dx = -\left(\tfrac{1}{3}x^3 - x^2\right)\Big|_0^2 = \tfrac{4}{3}.$$