**1.** Use the graph of the function $f$ below to determine the following where possible.

a. $\lim_{x \to -\infty} f(x)$

b. $\lim_{x \to \infty} f(x)$

c. $\lim_{x \to -3} f(x)$

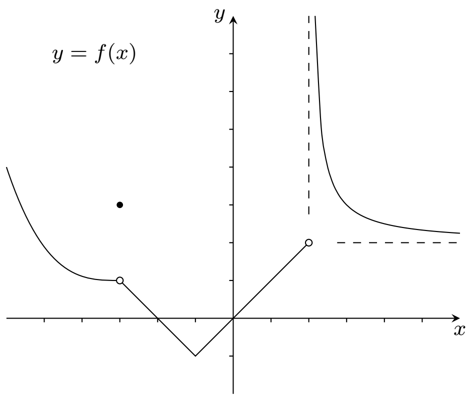
d. $\lim_{x \to 2^-} f(x)$

e. $\lim_{x \to 2} f(x)$

f. $\lim_{x \to -1} f(x)$

g. $\lim_{x \to 0^+} f(x)$

h. $f(-3)$




i. All values of $x$ at which $f$ is continuous but not differentiable.

**2.** Evaluate each of the following limits.

a. $\displaystyle\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^4 - 16}$ b. $\displaystyle\lim_{t \to \frac{1}{3}\pi^-} \frac{\sin t}{1 - 2\cos t}$ c. $\displaystyle\lim_{x \to 4} \frac{4 - x}{\sqrt{x + 5} - 3}$

d. $\displaystyle\lim_{x \to -\infty} \sqrt{\frac{2 + 9x}{5 + 4x}}$ e. $\displaystyle\lim_{x \to 2} \frac{x \log x - \log(x^2)}{x - 2}$

**3.** Discuss fully the continuity of $f$, where

$$f(x) = \begin{cases} |x - 2| & \text{if } x \leqslant 1, \\ \sqrt{x - 1} & \text{if } 1 < x < 5 \text{ and} \\ \dfrac{2x}{10 - x} & \text{if } x > 5 \text{ and } x \neq 10. \end{cases}$$

**4.** The function $f$ is defined by

$$f(x) = \begin{cases} x^p \sin(1/x) & \text{if } x > 0, \text{ and} \\ 0 & \text{if } x \leqslant 0, \end{cases}$$

where $p$ is a real number.
a. For which values of $p$ is $f$ continuous but not differentiable at 0?
b. For which values of $p$ is $f'$ continuous at 0?

**5.** Use the definition of derivative to find $f'(x)$, where $f(x) = \dfrac{2x}{7 - x}$.

**6.** Find $\dfrac{dy}{dx}$ for each of the following.

a. $y = (3x^2 - 9)\sec^4(5x + 3)$ b. $y = \log \dfrac{(x^2 - 9)^4}{x^3\sqrt{x + 7}}$ c. $y = e^{\sqrt{3 + \tan^2 x}}$

d. $y = \sin(2x - 1)^3 + \cos^3(2x - 1)$ e. $y = (\sin x)^{\log x}$

**7.** Find an equation of the tangent line to the graph of

$$y = \frac{\sin x}{1 + \cos x}$$

at the point where $x = \frac{2}{3}\pi$.

**8.** Given

$$f(x) = \frac{x^2 + 1}{(x + 1)^2}.$$

a. Find $f'(x)$ and simplify your answer as much as possible.
b. Determine whether $f$ has a local maximum or minimum at $x = 1$. You may use the fact that

$$f''(x) = \frac{-4(x - 2)}{(x + 1)^4}.$$

**9.** Given the curve with equation $x^2 - xy + y^2 = 9$.
a. Find $\dfrac{dy}{dx}$.
b. Find all point on the curve where the tangent line is horizontal;
c. A point $P(x, y)$ moves along the curve so that $\frac{dx}{dt} = -2$ when $x = 0$ and $y = 3$. At what rate (and in which direction) is the line tangent to the curve at $P$ rotating at this instant?

**10.** Given

$$f(x) = \frac{x^3 - 8}{x^3 + 8}, \qquad f'(x) = \frac{48x^2}{(x^3 + 8)^2} \quad \text{and} \quad f''(x) = \frac{-192x(x^3 - 4)}{(x^3 + 8)^3},$$

sketch the graph of $f$. Make sure your solution includes all (if any) intercepts, asymptotes, intervals of monotonicity, extrema, intervals of concavity and points of inflection.

**11.** Find the absolute extrema of $f(x) = x^{1/3} - x^{2/3}$ on $[-1, 1]$.

**12.** Find the intervals of monotonicity and concavity of $f$, where $f(x) = x^2 e^x$.

**13.** Let $\vartheta$ be the radian measure of an angle in a right-angled triangle and let $x$ and $y$ be, respectively, the lengths of the sides adjacent to and opposite to that angle. Suppose that $x$ and $y$ vary with time. At the instant when $x$ is 2 and is increasing at 4 units per second, $y$ is 2 and is decreasing at 1 unit per second. At what rate is $\vartheta$ changing at this instant?

**14.** Michael has 340 m of fencing to enclose two separate fields, one of which will be a rectangle twice as long as it is wide, and the other will be a square. Due to zoning regulations, the width of the rectangular field has to be at least 20 m and at most 50 m. Find the maximum and minimum total area.

**15.** Find an expression (or the exact value of) each of the following limits.

a. $\displaystyle\lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$ b. $\displaystyle\lim_{x \to 0^-} \frac{|x|}{x}$

c. $\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} e^{2i/n} \frac{2}{n}$ d. $\displaystyle\lim_{\vartheta \to 0} \frac{\sin \vartheta}{\vartheta}$

**16.** Show that the function $f$, defined by

$$f(x) = \sqrt{100 - x^2},$$

satisfies the hypotheses of the Mean Value Theorem on $[-6, 8]$. Then find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.

**17.** Evaluate each of the following integrals.

a. $\displaystyle\int_1^2 \frac{(2y + 1)^2}{y}\, dy$ b. $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \frac{\sec \vartheta + \tan \vartheta}{\cos \vartheta}\, d\vartheta$ c. $\displaystyle\int (1 - x + x^2)\sqrt{x}\, dx$

**18.** Verify the integral formula $\displaystyle\int \frac{4}{3(4 - 3x^2)^{3/2}}\, dx = \frac{x}{3\sqrt{4 - 3x^2}} + C.$

**19.** Find $z$ given that

$$\frac{d^2 z}{dt^2} = 2\sin t - 4\cos t + e^t, \qquad \left.\frac{dz}{dt}\right|_{t=0} = 3 \quad \text{and} \quad z(0) = -5.$$

**20.** Let $S$ be the region bounded by the graph of $f(x) = \dfrac{3 + x}{x}$ and the $x$-axis between $x = 1$ and $x = 9$.
a. Approximate the area of $S$ by finding the Riemann sum with four equal subintervals and taking midpoints as sample points.
b. What is the exact area of $S$?

**21.** Evaluate $\displaystyle\int_0^5 |x - 2|\, dx$ by interpreting it in terms of area.

**22.** Use the Fundamental Theorem of Calculus to find a function $f$ and a number $a$ such that

$$8 + \int_a^x \frac{f(t)}{t\sqrt{t}}\, dt = 4\log x.$$

**1.** Inspecting the graph of $f$ reveals that a. $\lim_{x\to-\infty} f(x) = \infty$, b. $\lim_{x\to\infty} f(x) = 2$, c. $\lim_{x\to-3} f(x) = 1$, d. $\lim_{x\to 2^-} f(x) = 2$, e. $\lim_{x\to 2} f(x)$ is undefined (since the left limit is 2 and the right limit is infinite), f. $\lim_{x\to-1} f(x) = -1$, g. $\lim_{x\to 0^+} f(x) = 0$, h. $f(-3) = 3$ and i. $f$ is continuous but not differentiable at $-1$.

**2.** a. Since the numerator and denominator both vanish at 2, they share the factor $x-2$. Independence and direct substitution then give

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+2x-8}{x^4-16} &= \lim_{x\to 2}\frac{(x-2)(x+4)}{(x-2)(x+2)(x^2+4)}\\ &= \lim_{x\to 2}\frac{x+4}{(x+2)(x^2+4)}\\ &= \tfrac{3}{16}.\end{aligned}$$

b. Since $\sin t \to \frac12\sqrt{3}$ and $1-2\cos t \to 0^-$ as $t \to \frac13\pi^-$, it follows that

$$\lim_{t\to\frac13\pi^-}\frac{\sin t}{1-2\cos t} = -\infty.$$

c. Rationalizing the denominator gives

$$\begin{aligned}\lim_{x\to 4}\frac{4-x}{\sqrt{x+5}-3}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} &= \lim_{x\to 4}\frac{(4-x)(\sqrt{x+5}+3)}{x-4}\\ &= -\lim_{x\to 4}(\sqrt{x+5}+3)\\ &= -6,\end{aligned}$$

by independence and direct substitution.

d. Extracting dominant powers of $x$ gives

$$\lim_{x\to-\infty}\sqrt{\frac{2+9x}{5+4x}} = \lim_{x\to-\infty}\sqrt{\frac{2/x+9}{5/x+4}} = \tfrac32,$$

since $2/x \to 0$ and $5/x \to 0$ as $x \to -\infty$.

e. Writing $\log(x^2) = 2\log x$ and factoring gives

$$\lim_{x\to 2}\frac{x\log x - \log(x^2)}{x-2} = \lim_{x\to 2}\frac{(x-2)\log x}{x-2} = \lim_{x\to 2}\log x = \log 2,$$

by independence and direct substitution.

**3.** Each part of $f$ is defined by a function continuous at every real number in its domain, so $f$ is continuous at every real number except possibly 1, 5 and 10. Since

$$f(1) = \lim_{x\to 1^-} f(x) = 1 \quad\text{and}\quad \lim_{x\to 1^+} f(x) = 0,$$

it follows that $f$ has a jump discontinuity at 1. Next, $f(5)$ is undefined, but

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 2,$$

so $f$ has a removable discontinuity at 5. Finally, $f(10)$ is undefined, and

$$\lim_{x\to 10^\pm} f(x) = \mp\infty,$$

so $f$ has an infinite discontinuity at 10.

**4.** a. If $p \leqslant 0$ then, since $|x^p\sin(1/x)| \geqslant |\sin(1/x)|$ for $0 < x < 1$, and since $\sin(1/x)$ oscillates between $-1$ and 1 as $x \to 0$, it follows that $f$ is discontinuous at 0. On the other hand, if $p > 0$ then $|x^p| \to 0$ as $x \to 0$ and $|x^p\sin(1/x)| \leqslant |x^p|$, so

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

Thus, $f$ is continuous at 0 if, and only if, $p > 0$. Next,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = 0,$$

and by the foregoing argument

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} x^{p-1}\sin(1/x)$$

exists (and is equal to 0) if, and only if, $p-1 > 0$, *i.e.*, $p > 1$, in which case $f'(0) = 0$. So $f$ is continuous but not differentiable at 0 if, and only if, $0 < p \leqslant 1$.

b. By the solution to the first part of this problem, $f$ is differentiable at 0 if, and only if, $p > 1$, in which case $f'(0) = 0$. Notice also that $f'(x) = 0$ if $x < 0$, so $f'$ is continuous from the left at 0. On the other hand, if $x > 0$ then by the Product Rule and the Chain Rule,

$$\begin{aligned}f'(x) &= px^{p-1}\sin(1/x) + x^p\cos(1/x)\cdot(-1/x^2)\\ &= px^{p-1}\sin(1/x) - x^{p-2}\cos(1/x).\end{aligned}$$

By the argument from part a of this question,

$$\lim_{x\to 0^+} px^{p-1}\sin(1/x)$$

is zero if $p > 1$ or $p = 0$, and undefined if $p \leqslant 1$ and $p \neq 0$, and

$$\lim_{x\to 0^+} x^{p-2}\cos(1/x)$$

is zero if $p > 2$ and is undefined if $p \leqslant 2$. This implies, since $|\sin(1/x)| = 1$ when $\cos(1/x) = 0$ and $|\cos(1/x)| = 1$ when $\sin(1/x) = 0$, that $f'$ is continuous at 0 if, and only if, $p > 2$.

**5.** By the definition of the derivative of a function,

$$\begin{aligned}f'(x) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\ &= \lim_{t\to x}\frac{\dfrac{2t}{7-t}-\dfrac{2x}{7-x}}{t-x}\\ &= 2\lim_{t\to x}\frac{t(7-x)-x(7-t)}{(t-x)(7-t)(7-x)}\\ &= 14\lim_{t\to x}\frac{t-x}{(t-x)(7-t)(7-x)}\\ &= 14\lim_{t\to x}\frac{1}{(7-t)(7-x)}\\ &= \frac{14}{(7-x)^2}.\end{aligned}$$

**6.** a. By the Product Rule and the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= 6x\sec^4(5x+3) + 3(x^2-3)4\sec^4(5x+3)\tan(5x+3)\cdot 5\\ &= 6\sec^4(5x+3)\big(x + 10(x^2-3)\tan(5x+3)\big).\end{aligned}$$

b. If

$$y = \log\frac{(x^2-9)^4}{x^3\sqrt{x+7}} = 4\log(x^2-9) - 3\log x - \tfrac12\log(x+7)$$

then by the linearity of the derivative and the Chain Rule,

$$\frac{dy}{dx} = \frac{4\cdot 2x}{x^2-9} - \frac{3}{x} - \frac{1}{2(x+7)} = \frac{9x^3+70x^2+63x+378}{2x(x+7)(x^2-9)}.$$

c. By the Chain Rule,

$$\frac{dy}{dx} = \frac{e^{\sqrt{3+\tan^2 x}}\,2\tan x\sec^2 x}{2\sqrt{3+\tan^2 x}} = \frac{e^{\sqrt{3+\tan^2 x}}\,\sin x}{\sqrt{3+\tan^2 x}\,\cos^3 x}.$$

d. By the linearity of the derivative and the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \cos(2x-1)^3\cdot 3(2x-1)^2\cdot 2 + 3\cos^2(2x-1)\cdot\big(-\sin(2x-1)\big)\cdot 2\\ &= 6\big((2x-1)^2\cos(2x-1)^3 - \cos^2(2x-1)\sin(2x-1)\big).\end{aligned}$$

e. If $y = (\sin x)^{\log x} = e^{\log x\log(\sin x)}$, then by the Product Rule and the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= e^{\log x\log(\sin x)}\cdot\big\{(1/x)\log(\sin x) + (\log x)(\cos x/\sin x)\big\}\\ &= (\sin x)^{\log x}\big(\log(\sin x) + x\log x\cot x\big)/x.\end{aligned}$$

**7.** If $x = \frac23\pi$ then $y = (\sin\frac23\pi)/(1+\cos\frac23\pi) = \frac12\sqrt{3}/(1-\frac12) = \sqrt{3}$. Next, by the Quotient Rule,

$$\frac{dy}{dx} = \frac{(\cos x)(1+\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} = \frac{1}{1+\cos x},$$

and hence the slope of the tangent line in question is equal to

$$\left.\frac{dy}{dx}\right|_{x=\frac23\pi} = \frac{1}{1-\frac12} = 2.$$

Therefore, an equation of the line tangent to the given curve at the point where $x = \frac{2}{3}\pi$ is $y = \sqrt{3} + 2\big(x - \frac{2}{3}\pi\big)$.

**8.** a. Writing $f(x) = (x^2+1)(x+1)^{-2}$, and applying the Product Rule and the Chain Rule, gives

$$\begin{aligned} f'(x) &= 2x(x+1)^{-2} - 2(x^2+1)(x+1)^{-3} \\ &= 2\big(x(x+1) - (x^2+1)\big)(x+1)^{-3} \\ &= 2(x-1)(x+1)^{-3}. \end{aligned}$$

b. Since $f'(1) = 0$ and $f''(1) = -4(-3)2^{-4} = \frac{3}{4} > 0$, it follows that $f(1) = \frac{1}{2}$ is a local minimum value of $f$.

**9.** a. Where $y$ is an implicit function of $x$ defined by the equation $x^2 - xy + y^2 = 9$, the Chain Rule gives

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{y-2x}{2y-x}.$$

b. The tangent line to the graph of the given equation is horizontal where $\frac{dy}{dx} = 0$, *i.e.*, where $y = 2x$ (and $2y \neq x$). Replacing $y$ by $2x$ in the given equation yields $x^2 - x(2x) + (2x)^2 = 9$, or $x^2 = 3$, which gives $x = \pm\sqrt{3}$ and $y = \pm 2\sqrt{3}$. Therefore, the tangent line to the graph of $x^2 - xy + y^2 = 9$ is horizontal at the points $(\pm\sqrt{3}, \pm 2\sqrt{3})$.

c. The angle $\alpha$ between the tangent line to the curve at a given point and the positive $x$ axis satisfies $\tan\alpha = \frac{dy}{dx}$, and therefore

$$\sec^2\alpha\,\frac{d\alpha}{dx} = \frac{d^2y}{dx^2}, \qquad \text{or} \qquad \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{d\alpha}{dx} = \frac{d^2y}{dx^2}.$$

Now

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{\left(\frac{dy}{dx} - 2\right)(2y-x) - (y-2x)\left(2\frac{dy}{dx} - 1\right)}{(2y-x)^2} \\ &= \frac{3\left(x\frac{dy}{dx} - y\right)}{(2y-x)^2} \\ &= \frac{3}{(2y-x)^2}\left\{x \cdot \frac{y-2x}{2y-x} - y\right\} \\ &= \frac{-6(x^2 - xy + y^2)}{(2y-x)^3} \\ &= \frac{-54}{(2y-x)^3}, \end{aligned}$$

and so

$$\left.\frac{dy}{dx}\right|_{\substack{x=0,\\ y=3}} = \frac{1}{2} \qquad \text{and} \qquad \left.\frac{d^2y}{dx^2}\right|_{\substack{x=0,\\ y=3}} = -\frac{1}{4},$$

and therefore

$$\left(1 + \left(\tfrac{1}{2}\right)^2\right)\frac{d\alpha}{dx} = -\tfrac{1}{4}, \qquad \text{or} \qquad \frac{d\alpha}{dx} = -\tfrac{1}{5}.$$

Finally, by the Chain Rule,

$$\frac{d\alpha}{dt} = \frac{d\alpha}{dx}\frac{dx}{dt} = \left(-\tfrac{1}{5}\right)(-2) = \tfrac{2}{5},$$

so the line tangent to the given curve at $P$ is rotating (counterclockwise) at a rate of $\frac{2}{5}$ radians per unit of time when $P$ is $(0,3)$.

**10.** Factorizing the denominator of $f(x)$ gives $(x+2)(x^2-2x+4)$, and since $x^2 - 2x + 4 = (x-1)^2 + 3$ is always positive, the domain of $f$ is $\mathbb{R} \setminus \{-2\}$. The $y$ intercept of the graph of $f$ is $(0,-1)$. Factorizing the numerator of $f(x)$ gives $(x-2)(x^2+2x+4)$, and since $x^2+2x+4 = (x+1)^2+3$ is always positive, the $x$ intercept of the graph of $f$ is $(2,0)$. Next, since

$$\lim_{x\to -2^\pm} f(x) = \lim_{x\to -2^\pm}\frac{(x-2)(x^2+2x+4)}{(x+2)(x^2-2x+4)} = \mp\infty,$$

the line defined by $x = -2$ is the vertical asymptote of the graph of $f$. Also,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{1 - 8/x^3}{1 + 8/x^3} = 1,$$

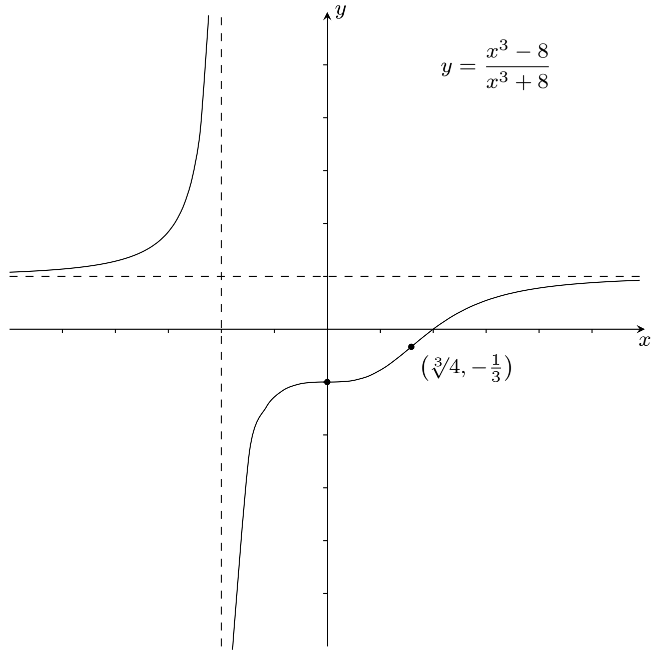
so the line defined by $y = 1$ is the horizontal asymptote of the graph of $f$. Where defined,

$$f'(x) = \frac{48x^2}{(x^3+8)^2}$$

is positive or zero, and is zero only if $x = 0$, so $f$ is increasing on $(-\infty, -2)$ and on $(-2, \infty)$. Therefore, $f$ has no local extrema. Finally,

$$f''(x) = \frac{-192x(x^3-4)}{(x^3+8)^3}$$

is zero if, and only if, $x$ is 0 or $\sqrt[3]{4}$, and is defined on the domain of $f$. Examining the sign of $f''$ between $-2$, 0 and $\sqrt[3]{4}$ reveals that the graph of $f$ is concave up on $(-\infty, -2)$ and on $(0, \sqrt[3]{4})$, and concave down on $(-2, 0)$ and on $(\sqrt[3]{4}, \infty)$, with points of inflection at $(0,-1)$ and $\big(\sqrt[3]{4}, -\frac{1}{3}\big)$. Below is a sketch of the graph of $f$, with unit lengths marked along the coordinate axes, asymptotes drawn as dashed lines, and the points of inflection emphasized.




**11.** Since $f$ is continuous on $\mathbb{R}$ and

$$f'(x) = \tfrac{1}{3}x^{-2/3} - \tfrac{2}{3}x^{1/3} = \tfrac{1}{3}x^{-2/3}(1 - 2\sqrt[3]{x}),$$

the critical number of $f$ are 0 and $\frac{1}{8}$, each of which belongs to $[-1, 1]$. Since

$$f(-1) = -2, \qquad f(0) = 0, \qquad f\left(\tfrac{1}{8}\right) = \tfrac{1}{4} \qquad \text{and} \qquad f(1) = 0,$$

the largest and smallest values of $f$ on $[-1,1]$ are $f\left(\frac{1}{8}\right) = \frac{1}{4}$ and $f(-1) = -2$.

**12.** If $f(x) = x^2e^x$ then by the Product Rule,

$$f'(x) = 2xe^x + x^2e^x = (x^2+2x)e^x = x(x+2)e^x,$$

so the critical numbers of $f$ are $-2$, 0. Since $e^x$ is positive for all real values of $x$, the sign of $f'(x)$ is the same as that of $x(x+2)$, which is positive if $x < -2$ or $x > 0$ and negative if $0 < x < 2$. Therefore, $f$ is increasing on $(-\infty, -2)$ and on $(0, \infty)$, and decreasing on $(-2, 0)$. Next,

$$f''(x) = (2x+2)e^x + (x^2+2x)e^x = (x^2+4x+2)e^x$$

and, since $e^x$ is positive for all real values of $x$, $f''(x)$ is positive, negative or zero with $x^2+4x+2 = (x+2)^2 - 2$, which is zero if $x = -2 \pm \sqrt{2}$, positive if $x < -2 - \sqrt{2}$ or $x > -2 + \sqrt{2}$, and negative if $-2 - \sqrt{2} < x < -2 + \sqrt{2}$. So the graph of $f$ is concave up on $(-\infty, -2-\sqrt{2})$ and on $(-2+\sqrt{2}, \infty)$, and concave down on $(-2-\sqrt{2}, -2+\sqrt{2})$.

**13.** Since $x\tan\vartheta = y$, differentiating with respect to time yields

$$\frac{dx}{dt}\tan\vartheta + x\sec^2\vartheta\,\frac{d\vartheta}{dt} = \frac{dy}{dt}, \qquad \text{or} \qquad \frac{dx}{dt}\tan\vartheta + x(1+\tan^2\vartheta)\frac{d\vartheta}{dt} = \frac{dy}{dt}.$$

At the instant when $x = 2$ and $y = 2$, $\tan\vartheta = 1$, $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = -1$, and so

$$4 + 2\big(1+1^2\big)\frac{d\vartheta}{dt} = -1, \qquad \text{or} \qquad \frac{d\vartheta}{dt} = -\tfrac{5}{4}.$$

Therefore, $\vartheta$ is changing at a rate of $-\frac{5}{4}$ radians per second at the instant when $x = 2$ and $y = 2$.

**14.** If $w$ denotes the width of the rectangle and $s$ denotes the side of the square then $6w + 4s = 340$, or $s = \frac{1}{2}(170 - 3w)$, so the total area of the fields is $A = 2w^2 + s^2 = 2w^2 + \frac{1}{4}(170 - 3w)^2$, and it is required to find the smallest and largest values of $A$ on the closed interval $[\,20, 50\,]$. Now

$$\frac{dA}{dw} = 4w - \tfrac{3}{2}(170 - 3w) = \tfrac{17}{2}(w - 30),$$

so the critical number 30 of $A$ lies in the interval $(\,20, 50\,)$, and comparing

$$A\Big|_{w=20} = 3825, \qquad A\Big|_{w=30} = 3400 \qquad \text{and} \qquad A\Big|_{w=50} = 5100,$$

reveals that the smallest total area is 3400 square metres and the largest total area is 5100 square metres.

**15.** a. $\displaystyle\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \frac{d}{dx}(\sin x) = \cos x.$

b. $\displaystyle\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}(-1) = -1.$

c. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} e^{2i/n}\,\frac{2}{n} = \int_0^2 e^x\,dx = e^x\Big|_0^2 = e^2 - 1.$

d. $\displaystyle\lim_{\vartheta\to 0}\frac{\sin\vartheta}{\vartheta} = 1.$

**16.** $f$ is continuous on $[\,-10, 10\,]$ and differentiable on $(\,-10, 10\,)$, so it certainly satisfies the hypotheses of the Mean Value Theorem on $[\,-6, 8\,]$. Therefore, there is a real number $\xi$ in $(\,-6, 8\,)$ such that $f(8) - f(-6) = f'(\xi)\big(8 - (-6)\big) = 14f'(\xi)$. Now

$$f'(\xi) = \frac{-\xi}{\sqrt{100 - \xi^2}}, \qquad \text{and} \qquad f(8) - f(-6) = \sqrt{36} - \sqrt{64} = -2,$$

so $\xi$ satisfies $7\xi = \sqrt{100 - \xi^2}$, which implies that $50\xi^2 = 100$, or $\xi = \pm\sqrt{2}$. However, only $\sqrt{2}$ is a solution of $7\xi = \sqrt{100 - \xi^2}$, and so $\sqrt{2}$ is the only number in $(\,-6, 8\,)$ which satisfies the conclusion of the Mean Value Theorem.

**17.** a. Expanding, and then dividing and integrating term by term, gives

$$\begin{aligned}\int_1^2 \frac{(2y+1)^2}{y}\,dy &= \int_1^2 (4y + 4 + y^{-1})\,dy \\ &= (2y^2 + 4y + \log y)\Big|_1^2 \\ &= 10 + \log 2.\end{aligned}$$

b. Dividing and integrating term by term gives

$$\begin{aligned}\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \frac{\sec\vartheta + \tan\vartheta}{\cos\vartheta}\,d\vartheta &= \int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} (\sec^2\vartheta + \sec\vartheta\tan\vartheta)\,d\vartheta \\ &= (\tan\vartheta + \sec\vartheta)\Big|_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \\ &= \sqrt{3} - \sqrt{2} + 1.\end{aligned}$$

c. Expanding and integrating term by term gives

$$\begin{aligned}\int (1 - x + x^2)\sqrt{x}\,dx &= \int (x^{1/2} - x^{3/2} + x^{5/3})\,dx \\ &= \tfrac{2}{3}x^{3/2} - \tfrac{2}{5}x^{5/2} + \tfrac{2}{7}x^{7/2} + C.\end{aligned}$$

**18.** Applying the Product Rule and the Chain Rule gives

$$\begin{aligned}\frac{d}{dx}\left\{\frac{x}{3\sqrt{4 - 3x^2}}\right\} &= \frac{1}{3\sqrt{4 - 3x^2}} - \frac{x(-6x)}{2\cdot 3(4 - 3x^2)^{3/2}} \\ &= \frac{(4 - 3x^2) + 3x^2}{3(4 - 3x^2)^{3/2}} \\ &= \frac{4}{3(4 - 3x^2)^{3/2}},\end{aligned}$$

and therefore

$$\int \frac{4}{3(4 - 3x^2)^{3/2}}\,dx = \frac{x}{3\sqrt{4 - 3x^2}} + C,$$

as required.

**19.** By (the first form of) the Fundamental Theorem of Calculus

$$\begin{aligned}\frac{dz}{dt} &= 3 + \int_0^t (2\sin\tau - 4\cos\delta + e^\tau)\,d\tau \\ &= 4 - 2\cos t - 4\sin t + e^t,\end{aligned}$$

and

$$\begin{aligned}z &= -5 + \int_0^t (4 - 2\cos\tau - 4\sin\tau + e^\tau)\,d\tau \\ &= -10 - 4t - 2\sin t + 4\cos t + e^t.\end{aligned}$$

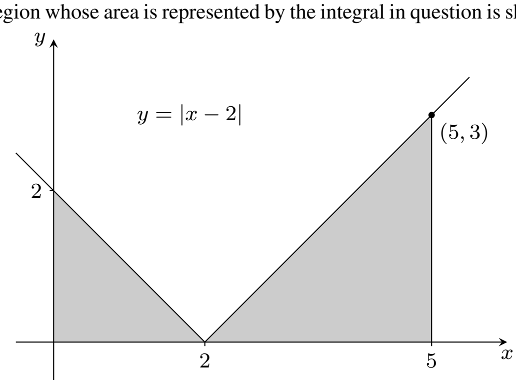
**20.** a. If $[\,1, 9\,]$ is partitioned into 4 subintervals of equal length then $\Delta x = 2$ and the midpoints of the subintervals are 2, 4, 6 and 8. The corresponding Riemann sum of $f(x) = (x+3)/x = 1 + 3/x$ is

$$2\big\{\big(1 + \tfrac{3}{2}\big) + \big(1 + \tfrac{3}{4}\big) + \big(1 + \tfrac{1}{2}\big) + \big(1 + \tfrac{3}{8}\big)\big\} = \tfrac{57}{4}.$$

b. The exact area of the region below the graph of $f$ and above the interval $[\,1, 9\,]$ on the $x$ axis is

$$\int_1^9 (1 + 3/x)\,dx = (x + 3\log x)\Big|_1^9 = 8 + 6\log 3.$$

**21.** The region whose area is represented by the integral in question is shaded below.




The region consists of two triangles. The triangle on the left has base 2, height 2 and area $\frac{1}{2}(2)(2) = 2$. The triangle on the right has base 3, height 3 and area $\frac{1}{2}(3)(3) = \frac{9}{2}$. Therefore,

$$\int_0^2 |x - 2|\,dx = 2 + \tfrac{9}{2} = \tfrac{13}{2}.$$

**22.** The given equation implies (taking $x = a$) that $8 = 4\log a$, *i.e.*, $\log a = 2$ or $a = e^2$. Next, if $f$ is continuous then differentiating and applying the Fundamental Theorem of Calculus gives

$$\frac{f(x)}{x\sqrt{x}} = \frac{4}{x}, \qquad \text{or} \qquad f(x) = 4\sqrt{x}.$$