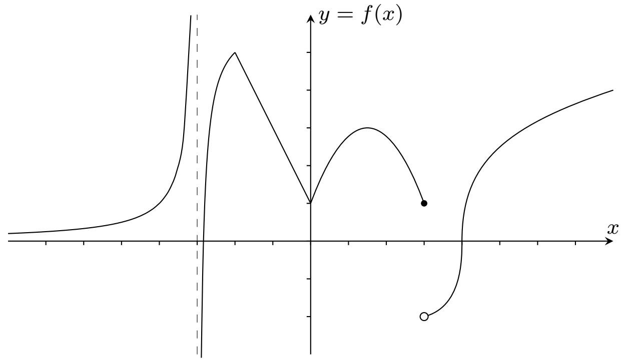
1. Given the the graph of f below, with unit lengths marked on the coordinate axes.



a. List the real numbers at which f is not differentiable.
b. List the real numbers at which f is continuous but not differentiable.

2. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 5} \frac{x^2 - 7x + 10}{3x - 15}$ b. $\displaystyle\lim_{x\to -\infty} \frac{6x}{\sqrt{9x^2 - 5x}}$ c. $\displaystyle\lim_{x\to 10} \frac{\frac{1}{5} - \frac{2}{x}}{x - 10}$

d. $\displaystyle\lim_{x\to 0} \frac{\sqrt{9 + 3x} - 3}{2x}$ e. $\displaystyle\lim_{x\to 0^-} \frac{\sin x - \frac{1}{\csc x} + e^x}{2\sec^2 x + \frac{|x|}{x}}$

3. Find all values of k such that the function f defined by

$$f(x) = \begin{cases} 3\cos(\pi x) - 10 & \text{if } x < 1, \text{ and} \\ kx - 6 & \text{if } x \geqslant 1, \end{cases}$$

is continuous on $\mathbb{R}$.

4. State and prove the Quotient Rule.

5. Does the function g, defined by $g(x) = x^3 + 3x^2 + 6x - 3$, have a zero in the interval $(0, 1)$? Justify your answer.

6. Use the definition of the derivative to find $f'(x)$, where $f(x) = \sqrt{3x - 1}$.

7. Find $\frac{dy}{dx}$ for each of the following.

a. $y = \sqrt[5]{x^3} + 5^x + \dfrac{1}{3x} + x^3 + \csc x - \sin \frac{1}{2}\pi$

b. $y = \sin^2(x^4 + 3) - \sec(\cot x)$ c. $y = (x^2 - 9)^{\sin x}$

d. $y = e^x \sec(3x^2 - 9)$ e. $y = \log \sqrt[3]{\dfrac{x}{(x+4)^5(x^2-9)}}$

8. Find an equation of the normal line to the graph of $y = x^3 - 4x - 3$ at the point where $x = -2$.

9. Where the equation $x^2 + xy + y^2 = 3$ defines y implicitly as a function of x,
a. find $\dfrac{dy}{dx}$,
b. determine all points at which the tangent line to the graph of the equation is parallel to the line defined by $y = x + 5$, and
c. find $\dfrac{d^2y}{dx^2}$ and simplify your answer as much as possible.

10. Determine all points on the graph of $y = x^3 \log(x^2)$ at which the tangent line is horizontal.

11. A paper cup has the shape of an inverted cone of height 20 cm and radius 5 cm. If water is poured into the cup at a rate of $3\,\text{cm}^3/\text{s}$, how fast is the water level rising when the water is 2 cm deep?

12. Sketch the graph of the function f defined by $f(x) = \dfrac{x}{4 - x^2}$.

13. Find the absolute maximum and absolute minimum values of the function f defined by $f(x) = x^{2/3} - x + 1$ on the closed interval $\left[-1, \frac{125}{8}\right]$.

14. A closed cylindrical can of volume $1\,\text{m}^3$ is to be designed to minimize cost. The cost of the top and bottom of the can is 80 cents per square metre, and the cost of the side is 50 cents per square metre. Find the dimensions of the least expensive can.

15. Find the position function of a moving particle moving horizontally whose acceleration is given by $a = 3t + 3\cos t$, for $t \geqslant 0$, whose initial velocity is 4 m/s and whose initial position is 3 m to the right of the origin.

16. Evaluate each of the following integrals.

a. $\displaystyle\int_1^e \frac{x - 3}{x}\, dx$ b. $\displaystyle\int \left(\sqrt[3]{x^2} + 2\sin x - e^2\right) dx$

c. $\displaystyle\int \left\{\sin^2 x + \cos^2 x - \frac{1}{\sec x}\right\} dx$

17. Find the area of the region bounded by the x-axis and the graphs of $y = \frac{1}{2}e^x$, $x = -1$ and $x = 3$.

18. Let f be the function defined by $f(x) = x^2 + 1$, and let $\mathscr{R}$ be the region below the graph of f and above the x-axis, over the interval $[0, 2]$.
a. Approximate the area of $\mathscr{R}$ using a Riemann sum with four subintervals of equal length, taking right endpoints for sample points.
b. Evaluate the exact area of $\mathscr{R}$ using a limit of Riemann sums.

19. Find the derivative with respect to x of $\displaystyle\int_{-3}^{x^3} e^{t^3}\, dt$.

1. a. f is not differentiable at -3 (which is not in its domain), -2, 0, 3 and 4.
b. f is continuous but not differentiable at -2, 0 and 4.

2. a. Since

$$\frac{x^2-7x+10}{3x-15}=\frac{(x-5)(x-2)}{3(x-5)}=\tfrac{1}{3}(x-2),$$

for $x\neq 5$, the limit in question is equal to

$$\lim_{x\to 5}\tfrac{1}{3}(x-2)=\tfrac{1}{3}(5-2)=1$$

by independence and direct substitution.
b. Since $\sqrt{x^2}=-x$ if $x<0$, extracting the dominant powers of x gives

$$\lim_{x\to-\infty}\frac{6x}{\sqrt{9x^2-5x}}=\lim_{x\to-\infty}\frac{-6}{\sqrt{9-5/x}}=\frac{-6}{\sqrt{9}}=-2$$

since $5/x\to 0$ as $x\to-\infty$.
c. Since

$$\frac{1}{5}-\frac{2}{x}=\frac{x-10}{5x},$$

one has (by independence and direct substitution)

$$\lim_{x\to 10}\frac{\dfrac{1}{5}-\dfrac{2}{x}}{x-10}=\lim_{x\to 10}\frac{1}{5x}=\tfrac{1}{50}.$$

d. Since

$$\frac{\sqrt{9+3x}-3}{2x}=\frac{3x}{2x\big(\sqrt{9+3x}+3\big)}=\frac{3}{2\big(\sqrt{9+3x}+3\big)}$$

for $x\neq 0$, it follows that the limit in question is equal to

$$\lim_{x\to 0}\frac{3}{2\big(\sqrt{9+3x}+3\big)}=\frac{3}{2(\sqrt{9}+3)}=\tfrac{1}{4}$$

by independence and direct substitution.
e. Since $\sin x=1/\csc x$ and $|x|/x=-1$ for $-\pi<x<0$, one has

$$\lim_{x\to 0^-}\frac{\sin x-\dfrac{1}{\csc x}+e^x}{2\sec^2 x+\dfrac{|x|}{x}}=\lim_{x\to 0^-}\frac{e^x}{2\sec^2 x-1}=\frac{e^0}{2\sec^2(0)-1}=1$$

by independence and direct substitution.

3. Since cosine function is continuous on $\mathbb{R}$, and linear combinations of continuous functions are continuous, f is continuous on $(-\infty,1)$. Also, polynomial functions are continuous on $\mathbb{R}$, so f is continuous on $[1,\infty)$. Therefore, f is continuous on $\mathbb{R}$ if, and only if,

$$f(1)=\lim_{x\to 1^-}f(x),\qquad\text{or}\qquad k-6=\lim_{x\to 1^-}\big(3\cos(\pi x)-10\big),$$

i.e., $k-6=-13$, which gives $k=-7$.

4. The Quotient Rule states that if the functions f and g are differentiable at the real number x and $g(x)$ is not equal to zero, then f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{\big(g(x)\big)^2}.$$

The Quotient Rule can be proved by an elementary calculation using the definition of the derivative, arithmetical limit laws and the fact that a differentiable function is continuous, as follows.

$$\begin{aligned}\left(\frac{f}{g}\right)'(x)&=\lim_{t\to x}\frac{(f/g)(t)-(f/g)(x)}{t-x}=\lim_{t\to x}\frac{f(t)g(x)-f(x)g(t)}{(t-x)g(t)g(x)}\\&=\frac{1}{\big(g(x)\big)^2}\lim_{t\to x}\frac{f(t)g(x)-f(x)g(t)}{t-x}\\&=\frac{1}{\big(g(x)\big)^2}\lim_{t\to x}\frac{f(t)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{t-x}\\&=\frac{1}{\big(g(x)\big)^2}\lim_{t\to x}\left\{\frac{f(t)-f(x)}{t-x}\cdot g(x)-f(x)\cdot\frac{g(t)-g(x)}{t-x}\right\}\\&=\frac{f'(x)g(x)-f(x)g'(x)}{\big(g(x)\big)^2}.\end{aligned}$$

5. Since g is a polynomial function it is continuous on $\mathbb{R}$ (and therefore continuous on the closed interval $[0,1]$), so the Intermediate Value Theorem implies that g has a zero in $(0,1)$, because $g(0)=-3<0<7=g(1)$.

6. If $f(x)=\sqrt{3x-1}$ then

$$\begin{aligned}f'(x)&=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\\&=\lim_{t\to x}\frac{\sqrt{3t-1}-\sqrt{3x-1}}{t-x}\cdot\frac{\sqrt{3t-1}+\sqrt{3x-1}}{\sqrt{3t-1}+\sqrt{3x-1}}\\&=\lim_{t\to x}\frac{3(t-x)}{(t-x)(\sqrt{3x-1}+\sqrt{3t-1})}\\&=\lim_{t\to x}\frac{3}{\sqrt{3x-1}+\sqrt{3t-1}}\\&=\frac{3}{2\sqrt{3x-1}}\end{aligned}$$

by independence and direct substitution.

7. a. $\dfrac{dy}{dx}=\tfrac{3}{5}x^{-2/5}+5^x\log 5-\tfrac{1}{3}x^{-2}+3x^2-\csc x\cot x$

b. $\dfrac{dy}{dx}=8x^3\sin(x^4+3)\cos(x^4+3)+\sec(\cot x)\tan(\cot x)\csc^2 x$

c. $\dfrac{dy}{dx}=y\dfrac{d}{dx}\{\log y\}=(x^2-9)^{\sin x}\left\{\dfrac{2x\sin x}{x^2-9}+\cos x\log(x^2-9)\right\}$

d. $\dfrac{dy}{dx}=e^x\sec^2(3x^2-9)\big(1+6x\tan(3x^2-9)\big)$

e. $\dfrac{dy}{dx}=\dfrac{1}{3x}-\dfrac{5}{3(x+4)}-\dfrac{2x}{3(x^2-9)}$

8. One has

$$y\Big|_{x=-2}=-3\qquad\text{and}\qquad\frac{dy}{dx}\Big|_{x=-2}=(3x^2-4)\Big|_{x=-2}=8,$$

so the equation of the normal line to the graph of $y=x^3-4x-3$ at the point $(-2,-3)$ is (since it is orthogonal to the tangent line) $y=-3-\frac{1}{8}(x+2)$, or $x+8y=-26$.

9. a. One has (by the Chain Rule, since y is a function of x)

$$2x+y+(x+2y)\frac{dy}{dx}=0,\qquad\text{or}\qquad\frac{dy}{dx}=-\frac{2x+y}{x+2y}.$$

b. The tangent line to the curve in question is parallel to the line defined by $y=x+5$ just in case

$$-\frac{2x+y}{x+2y}=1,\qquad\textit{i.e.},\qquad x+2y\neq 0\qquad\text{and}\qquad 2x+y=-x-2y.$$

The latter equation, *i.e.*, $y=-x$, combined with the equation defining the curve in question, yields $x^2=3$. So the tangents to the curve at the points $(\pm\sqrt{3},\mp\sqrt{3})$ are parallel to the given line (as are no other tangents).
c. Applying the Quotient Rule (and the Chain Rule) gives

$$\frac{d^2y}{dx^2}=-\frac{\left(2+\dfrac{dy}{dx}\right)(x+2y)-(2x+y)\left(1+2\dfrac{dy}{dx}\right)}{(x+2y)^2}.$$

Now

$$\left(2+\frac{dy}{dx}\right)(x+2y)=\left(2-\frac{2x+y}{x+2y}\right)(x+2y)=3y$$

and

$$(2x+y)\left(1+2\frac{dy}{dx}\right)=(2x+y)\left(1-\frac{4x+2y}{x+2y}\right)=-3x(2x+y),$$

so the numerator of the second derivative is

$$\frac{3y(x+2y)+3x(2x+y)}{x+2y}=\frac{6(x^2+xy+y^2)}{x+2y}=\frac{18}{x+2y}$$

since $x^2+xy+y^2=3$ on the given curve, and therefore

$$\frac{d^2y}{dx^2}=\frac{-18}{(x+2y)^3}.$$

10. If $y = x^3 \log(x^2) = 2x^3 \log|x|$ then

$$\frac{dy}{dx} = 6x^2 \log|x| + 2x^2 = 2x^2(3\log|x| + 1),$$

which is zero if, and only if, $\log|x| = -\frac{1}{3}$. The tangent line to the given curve is horizontal precisely where $\frac{dy}{dx} = 0$, *i.e.*, at the points $\big(\pm\sqrt[3]{e^{-1}}, \mp\frac{2}{3}e^{-1}\big)$.

11. If V, r and h denote, respectively, the volume, radius, and height of the part of the cup filled with water, then by similarity $r = \frac{1}{4}h$, so $V = \frac{1}{3}\pi r^2 h = \frac{1}{48}\pi h^3$, and therefore

$$\frac{dV}{dt} = \tfrac{1}{16}\pi h^2 \,\frac{dh}{dt}.$$

If the cup is being filled at a rate of $3\,\mathrm{cm}^3/\mathrm{s}$, then when the water is $2\,\mathrm{cm}$ deep,

$$3 = \tfrac{1}{4}\pi\,\frac{dh}{dt}, \qquad \text{or} \qquad \frac{dh}{dt} = 12/\pi,$$

i.e., the water level is rising by $12/\pi\,\mathrm{cm/s}$.

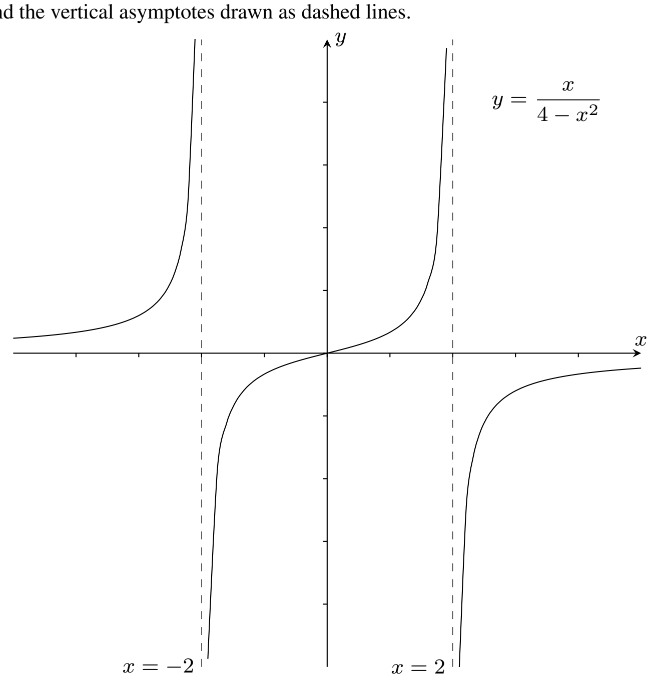
12. The domain of f is the set of all real numbers less ± 2, at which the graph of f has vertical asymptotes: $f(x) \to \infty$ as $x \to \pm 2^-$, and $f(x) \to -\infty$ $x \to \pm 2^+$. The origin is the only axis intercept of the graph of f. Since $f(x) \to 0$ as $x \to \pm\infty$, the x-axis is the horizontal asymptote of the graph of f. Since

$$f'(x) = \frac{4 - x^2 - x(-2x)}{(4 - x^2)^2} = \frac{x^2 + 4}{(x^2 - 4)^2},$$

f' is positive on its domain, which is same as the domain of f, so f is increasing on $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$, with no critical numbers (and therefore no local extrema). Also,

$$f''(x) = \frac{2x}{(x^2-4)^2} - \frac{4x(x^2+4)}{(x^2-4)^3} = -\frac{2x(x^2+12)}{(x^2-4)^3},$$

which is positive on $(-\infty, -2)$ and on $(0, 2)$, where the graph of f is concave up, and negative on $(-2, 0)$ and on $(2, \infty)$, where the graph of f is concave down. It follows that the origin is the only point of inflection of the graph of f. Follows a sketch of the graph of f, unit lengths marked along the coordinate axes and the vertical asymptotes drawn as dashed lines.



13. Since f is continuous on $\mathbb{R}$, and therefore continuous on $I = \big[-1, \frac{125}{8}\big]$, it assumes a largest value and a smallest value on I by the Extreme Value Theorem. Now

$$f'(x) = \tfrac{2}{3}x^{-1/3} - 1 = \frac{2 - 3\sqrt[3]{x}}{3\sqrt[3]{x}},$$

so the critical numbers of f are 0 (at which f' is undefined) and $\frac{8}{27}$ (at which f' vanishes), each of which belongs to the interior of I. Finally, comparing

$$f(-1) = 3, \qquad f(0) = 1, \qquad f\big(\tfrac{8}{27}\big) = \tfrac{31}{27} \qquad \text{and} \qquad f\big(\tfrac{125}{8}\big) = -\tfrac{67}{8},$$

reveals that the absolute maximum value of f on I is 3 and the absolute minimum value of f on I is $-\frac{67}{8}$.

14. If h denotes the height of the can and r denotes its radius (each measured in metres) then the cost of the can is given by

$$C = \tfrac{8}{5}\pi r^2 + \pi r h$$

dollars (where the first term is the cost of the top and bottom of the can, and the second term is the cost of its side). Since the volume of the can is $\pi r^2 h = 1$ cubic metre, it follows that the cost of the can as a function of its radius is given by

$$C = \tfrac{1}{5}\big(8\pi r^2 + 5r^{-1}\big)$$

dollars, for $r > 0$. Now

$$\frac{dC}{dt} = \tfrac{1}{5}\big(16\pi r - 5r^{-2}\big) = \tfrac{1}{5}r^{-2}\big(16\pi r^3 - 5\big),$$

which is defined on $(0, \infty)$ and vanishes at $\sqrt[3]{5/(16\pi)} = \frac{1}{2}\sqrt[3]{5/(2\pi)}$. Since

$$\frac{d^2C}{dt^2} = \tfrac{2}{5}\big(8\pi + 5r^{-3}\big) > 0$$

for $r > 0$, it follows that C has a global minimum value on $(0, \infty)$ at $\frac{1}{2}\sqrt[3]{5/(2\pi)}$. Therefore the least expensive such can has radius $\frac{1}{2}\sqrt[3]{5/(2\pi)}$ metres and height $4\sqrt[3]{4/(25\pi)}$ metres.

15. By the (first form of the) Fundamental Theorem of Calculus, the velocity function of the particle is

$$v = 4 + \int_0^t (3\tau + 3\cos\tau)\,d\tau = 4 + \tfrac{3}{2}t^2 + 3\sin t,$$

and the position of the particle is given by

$$s = 3 + \int_0^t \big(4 + \tfrac{3}{2}\tau^2 + 3\sin\tau\big)\,d\tau = 6 + 4t + \tfrac{1}{2}t^3 - 3\cos t.$$

16. a. Dividing and integrating term-by-term gives

$$\int_1^e \frac{x-3}{x}\,dx = \int_1^e (1 - 3/x)\,dx = \big(x - 3\log|x|\big)\Big|_1^e = e - 4.$$

b. Integrating term-by-term gives

$$\int \big(\sqrt[3]{x^2} + 2\sin x - e^2\big)\,dx = \tfrac{3}{5}x^{5/3} - 2\cos x - e^2 x + C.$$

c. Simplifying and then integrating term-by-term gives

$$\int \Big\{\sin^2 x + \cos^2 x - \frac{1}{\sec x}\Big\}\,dx = \int (1 - \cos x)\,dx = x - \sin x + C.$$

17. The area of the region in question is

$$\int_{-1}^3 \tfrac{1}{2}e^x\,dx = \tfrac{1}{2}e^x\Big|_{-1}^3 = \tfrac{1}{2}(e^3 - 1/e).$$

18. a. The partition of $[0, 2]$ into four subintervals of equal length has $\Delta x = \frac{1}{2}$ and $x_i = \frac{1}{2}i$, for $i = 0, 1, \ldots, 4$. The corresponding right endpoint Riemann sum is

$$\mathscr{R}_4 = \tfrac{1}{2}\big\{\big((\tfrac{1}{2})^2 + 1\big) + \big(1^2 + 1\big) + \big((\tfrac{3}{2})^2 + 1\big) + \big(2^2 + 1\big)\big\} = \tfrac{23}{4}.$$

b. The partition of $[0, 2]$ into n subintervals of equal length has $\Delta x = 2/n$ and $x_i = 2i/n$, for $i = 0, \ldots, n$. The corresponding right endpoint Riemann sum is

$$\begin{aligned}
\mathscr{R}_n &= \frac{2}{n}\sum_{i=1}^n \Big\{\Big(\frac{2i}{n}\Big)^2 + 1\Big\} = \frac{2}{n}\sum_{i=1}^n \Big\{\frac{4}{n^2}i^2 + 1\Big\} \\
&= \frac{2}{n}\Big\{\frac{4}{n^2}\cdot\tfrac{1}{6}n(n+1)(2n+1) + n\Big\} \\
&= \tfrac{4}{3}\big(1 + \tfrac{1}{n}\big)\big(2 + \tfrac{1}{n}\big) + 2.
\end{aligned}$$

Since f is continuous and positive on $[0, 2]$, it follows that the area between the graph of f and the x-axis on $[0, 2]$ is equal to

$$\int_0^2 (x^2 + 1)\,dx = \lim_{n\to\infty} \mathscr{R}_n = \tfrac{4}{3}\cdot 2 + 2 = \tfrac{14}{3}$$

(since $\frac{1}{n} \to 0$ as $n \to \infty$).

19. By the first form of the Fundamental Theorem of Calculus and the Chain Rule,

$$\frac{d}{dx}\int_{-3}^{x^3} e^{t^3}\,dt = e^{(x^3)^3}\cdot\frac{d}{dx}(x^3) = 3x^2 e^{x^9}.$$