1. Use the graph of the function f to determine each of the following. Where appropriate, use ∞, $-\infty$ or *undefined*.

a. $f(2)$

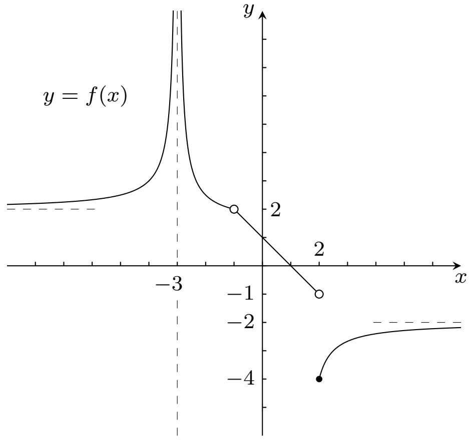
b. $\lim_{x\to -3} f(x)$

c. $\lim_{x\to -1} f(x)$

d. $\lim_{x\to 2^-} f(x)$

e. $\lim_{x\to 2} f(x)$

f. $\lim_{x\to -\infty} f(x)$



2. Evaluate the following. Use ∞, $-\infty$ or "undefined" where appropriate.

a. $\lim_{x\to 3} \dfrac{2x^2-5x-3}{x^2-8x+15}$ b. $\lim_{\vartheta\to 0} \dfrac{\vartheta^2-\vartheta}{\tan 4\vartheta}$

c. $\lim_{x\to\infty} \left\{\sqrt{x^2+5x}-\sqrt{x^2+2x}\right\}$ d. $\lim_{x\to 4^-} \dfrac{|x-4|}{(x-4)^2}$

e. $\lim_{x\to -2} f(x)$, given that $\dfrac{x^2-4}{x+2} \leqslant f(x) \leqslant x^2+5x+2$ for $x \neq -2$.

3. For which values of k, if any, is the limit

$$\lim_{x\to 5} \frac{x^2-3x+k}{x^2-4x-5}$$

defined? What is the value of the limit in each case?

4. Find and classify the discontinuities of the function f, defined by

$$f(x) = \begin{cases} \sqrt{1-x} & \text{if } x < -3, \\ \dfrac{x-3}{x} & \text{if } -3 < x \leqslant 4 \text{ and } x \neq 0, \text{ and} \\ \dfrac{x^2-16}{x^2-5x+4} & \text{if } x > 4. \end{cases}$$

5. Let $f(x) = \dfrac{4}{x-1}$.

a. Find all numbers which satisfy the conclusion of the Mean Value Theorem for f on the interval $[2,5]$.

b. Show that there is no real number which satisfies the conclusion of the Mean Value Theorem for f on the interval $[0,2]$. Why does this not contradict the Mean Value Theorem?

6. Use the definition of the derivative to find $f'(x)$, where $f(x) = \dfrac{x}{x+1}$.

7. Find $\dfrac{dy}{dx}$ for each of the following.

a. $y = 8x^7 + \sqrt[7]{x^8} - \log_8(x+7) + \dfrac{\sin(x^7)}{7} - 4^{3\pi} + e^{1/x}$

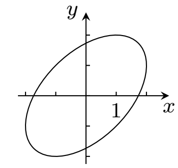
b. $y = \dfrac{(2x+1)^5}{x^2-3}$ c. $y = (x^3-1)^{\sec x}$

d. $y = \tan^3 x \csc(10x-1)$ e. $y = \log \dfrac{(4x-1)(x^2+1)^{3/2}}{e^{4x}\sqrt{x}}$

8. Find the 68th derivative of f, where $f(x) = 2^{2x} + \cos x - x^{67}$.

9. For which values of x is the tangent line to the graph of $y = (x-5)^4(2x-1)^5$ horizontal?

10. The graph of the equation $x^2 - xy + y^2 = 3$ is an ellipse, as displayed to the right. Find the points at which the ellipse crosses the x-axis, and show that the tangent lines at these points are parallel.

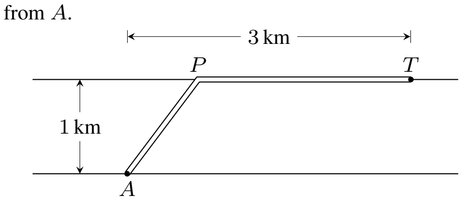


11. A spotlight on the ground shines on a wall 12 metres away. If a man 2 metres tall walks from the spotlight toward the wall at a speed of 1.6 metres per second, how fast is the height of his shadow on the wall changing when he is 4 metres from the wall?

12. Sketch the graph of the function f, defined by $f(x) = x\sqrt[3]{x+4}$.

13. Find the global extrema of $f(x) = (2x-1)\sqrt[3]{x}$ on $[-1,1]$.

14. An oil company has a refinery at point A on the bank of a straight river 1 km wide. It is going to run a pipe from point A to point P somewhere on the opposite side of a river, and then straight along the riverside to a tank T situated 3 km downstream from A.



It costs 15 thousand dollars per kilometre to run the pipe under the water and 9 thousand dollars per kilometre to run the pipe along the bank. What should be the distance from P to T in order to minimize the total cost of the pipe?

15. Evaluate each of the following integrals.

a. $\displaystyle\int \left(\frac{e^x}{4} + \frac{4^x}{e} + \frac{e}{4}\right) dx$ b. $\displaystyle\int (2+\sin x)\sec^2 x \, dx$

c. $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (a\sin x + b\cos x)\, dx$

16. Find $f(x)$, for $x \neq 0$, if $f'(x) = \dfrac{2x^2-3x+4}{x}$, $f(1) = 0$ and $f(-1) = 3$.

17. Sketch and shade the region bounded by the x-axis and the graph of $y = \cos x$ from $x = 0$ to $x = \frac{3}{2}\pi$. Find the area of the region.

18. Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{2i}{n}\right) e^{\frac{2i}{n}} \left(\tfrac{2}{n}\right)$$

as a definite integral, and evaluate it directly (*i.e.*, without using the Fundamental Theorem of Calculus).

19. Suppose that f is a continuous function such that

$$\int_1^9 f(t)\, dt = 4,$$

and let

$$F(x) = \int_1^{x^2} f(t)\, dt.$$

Find: a. $F(1)$, b. $F(3)$ and c. $F'(x)$.

1. Directly from the graph of f, one has a. $f(2) = -4$, b. $\lim_{x\to-3} f(x) = \infty$, c. $\lim_{x\to-1} f(x) = 2$, d. $\lim_{x\to 2^-} f(x) - 1$, e. $\lim_{x\to 2} f(x)$ is undefined (since the left and right limits as $x \to 2$ of f are different) and f. $\lim_{x\to-\infty} f(x) = 2$.

2. a. Cancelling the common factor $x-3$ from the numerator and denominator gives, by independence and direct substitution,

$$\lim_{x\to3}\frac{2x^2-5x-3}{x^2-8x+15} = \lim_{x\to3}\frac{2x+1}{x-5} = -\tfrac{7}{2}.$$

b. Arithmetical properties of limits give

$$\lim_{\vartheta\to0}\frac{\vartheta^2-\vartheta}{\tan 4\vartheta} = \tfrac14 \lim_{\vartheta\to0}\big\{(\vartheta-1)\cos4\vartheta\big\}\left(\lim_{\vartheta\to0}\frac{\sin4\vartheta}{4\vartheta}\right)^{-1} = -\tfrac14,$$

by direct substitution and the fact that $(\sin t)/t \to 1$ as $t\to0$.

c. Rationalizing the numerator and extracting dominant powers of x gives (since $\sqrt{x^2} = x$ if $x>0$)

$$\lim_{x\to\infty}\big\{\sqrt{x^2+5x}-\sqrt{x^2+2x}\big\} = \lim_{x\to\infty}\frac{3}{\sqrt{1+5/x}+\sqrt{1+2/x}} = \tfrac32,$$

since $1/x\to0$ as $x\to\infty$.

d. Since $(x-4)^2 = |x-4|^2$, one has

$$\lim_{x\to4^-}\frac{|x-4|}{(x-4)^2} = \lim_{x\to4^-}\frac{1}{|x-4|} = \infty,$$

since $|x-4|\to0^+$ as $x\to4^-$.

e. Since

$$\frac{x^2-4}{x+2} \leqslant f(x) \leqslant x^2+5x+2 \qquad \text{if} \qquad x\neq-2,$$

$$\lim_{x\to-2}\frac{x^2-4}{x+2} = \lim_{x\to-2}(x-2) = -4,$$

and

$$\lim_{x\to-2}(x^2+5x+2) = -4,$$

it is plain that $\lim_{x\to-2} f(x) = -4$.

3. Since the denominator of the expression in the limit vanishes as $x\to5$, the limit will be undefined unless the numerator also vanishes as $x\to5$; *i.e.*, unless $5^2-3\cdot5+k=0$, or $k=-10$. In this case, the numerator and denominator factorize, respectively, as $(x-5)(x+2)$ and $(x-5)(x+1)$, and so the limit in question is equal to

$$\lim_{x\to5}\frac{x+2}{x+1} = \tfrac76$$

by independence.

4. Since an algebraic function is continuous on any interval in its domain, f is continuous on $(-\infty,-3)$, $(-3,0)$, $(0,4]$ and $(4,\infty)$. Now

$$\lim_{x\to-3^-} f(x) = 2 \qquad\text{and}\qquad \lim_{x\to-3^+} f(x) = 2,$$

but $f(-3)$ is undefined, so f has a removable discontinuity at -3. Next,

$$\lim_{x\to0^-} f(x) = \infty \qquad\text{and}\qquad \lim_{x\to0^+} f(x) = -\infty,$$

so f has an infinite discontinuity at 0. Finally,

$$\lim_{x\to4^+} f(x) = \lim_{x\to4^+}\frac{x+4}{x-1} = \tfrac83$$

by independence, and

$$\lim_{x\to4^-} f(x) = f(4) = \tfrac14,$$

so f has a jump discontinuity at 4.

5. a. The function f is differentiable on $(-\infty,1)$ and on $(1,\infty)$, so it satisfies the hypothesis of the Mean Value Theorem on $[2,5]$, and there is a real number ξ in $(2,5)$ such that $f(5)-f(2) = f'(\xi)(5-2)$, or $f'(\xi) = -1$, *i.e.*,

$$\frac{-4}{(\xi-1)^2} = -1, \qquad\text{or}\qquad (\xi-1)^2 = 4,$$

in $(2,5)$. By inspection (or by taking square roots and discarding the extraneous solution), one finds that the only such number is 3.

b. Here $f(2)-f(0) = 4-(-4) = 8$ so ξ will satisfy the conclusion of the Mean Value Theorem for f on $[0,2]$ if, and only if, ξ belongs to $(0,2)$ and $f'(\xi) = 4$, *i.e.* (as in Part a), $(\xi-1)^2 = -1$, which has no solutions. Since f has an infinite discontinuity at 1, f is not continuous on $[0,2]$, so the Mean Value Theorem does not apply in this case.

6. One has

$$f(t)-f(x) = \frac{t}{t+1}-\frac{x}{x+1} = \frac{t-x}{(t+1)(x+1)}$$

provided neither t nor x is -1, and so

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{t\to x}\frac{1}{(t+1)(x+1)} = \frac{1}{(x+1)^2}$$

for $x\neq-1$ by independence and direct substitution.

7. a. $\dfrac{dy}{dx} = 56x^6 + \tfrac87\sqrt[7]{x} - \dfrac{1}{(x+7)\log 8} + x^6\cos(x^7) - x^{-2}e^{1/x}$

b. $\dfrac{dy}{dx} = \dfrac{10(2x+1)^4}{x^2-3} - \dfrac{2x(2x+1)^5}{(x^2-3)^2} = \dfrac{2(2x+1)^4(3x^2-x-15)}{(x^2-3)^2}$

c. $\dfrac{dy}{dx} = y\dfrac{d}{dx}\{\log y\}$
$= (x^3-1)^{\sec x-1}(\sec x)\big((x^3-1)\tan x\,\log(x^3-1)+3x^2\big)$

d. $\dfrac{dy}{dx} = \tan^2 x\,\csc(10x-1)\big(3\sec^2 x - 10\tan x\,\cot(10x-1)\big)$

e. $\dfrac{dy}{dx} = \dfrac{4}{4x-1}+\dfrac{3x}{x^2+1}-\dfrac{1}{2x}-4$

8. Using the derivatives of exponential and power functions, one has

$$\frac{d^{68}}{dx^{68}}\big(2^{2x}\big) = 2^{2x}(2\log2)^{68} = 2^{2(x+34)}(\log2)^{68} \quad\text{and}\quad \frac{d^{68}}{dx^{68}}\big(x^{67}\big) = 0.$$

Also, $(d/dx)^4(\cos x) = \cos x$, and $68 = 17\cdot4$, so $(d/dx)^{68}(\cos x) = \cos x$. Therefore, $f^{(68)}(x) = 2^{2(x+34)}(\log2)^{68}+\cos x$.

9. The tangent line to the graph of the given equation is horizontal where

$$\begin{aligned}\frac{dy}{dx} &= 4(x-5)^3(2x-1)^5+10(x-5)^4(2x-1)^4\\ &= 18(x-5)^3(2x-1)^4(x-3)\end{aligned}$$

is equal to zero; *i.e.*, where x is $\tfrac12$, 3 or 5.

10. The ellipse meets the x-axis where $y=0$; *i.e.*, $x^2=3$. So the x-intercepts of the ellipse are $(\pm\sqrt3,0)$. Now by the Chain Rule (and the Implicit Function Theorem)

$$2x-y+(2y-x)\frac{dy}{dx} = 0, \qquad\text{or}\qquad \frac{dy}{dx} = \frac{y-2x}{2y-x},$$

provided $y\neq2x$, so the slope of the tangent line to the ellipse at each of its x-intercepts is 2; *i.e.*, the tangent lines are parallel, as required.

11. If x denotes the distance between the man and the spotlight and y denotes the height of his shadow on the wall (each measured in metres), then by similarity of triangles $x/2 = 12/y$, or $xy = 24$, and so

$$\frac{dx}{dt}y + x\frac{dy}{dt} = 0, \qquad\text{or}\qquad \frac{dy}{dt} = -\tfrac85 yx^{-1} = -\tfrac{192}{5}x^{-2},$$

provided $x\neq0$, since $\frac{dx}{dt} = \frac85$. When the man is four metres from the wall $x=8$, and so $\frac{dy}{dt} = -\frac{192}{5\cdot64} = -\frac35$. Therefore, when the man is four metres from the wall the height of his shadow is decreasing by $\frac35$ metres per second.

12. The function f is continuous on $\mathbb{R}$, and $f(x) = x^{4/3}(1+4/x)^{1/3}$ if $x\neq0$, so the graph of f has no horizontal, vertical or oblique asymptotes. The intercepts of the graph of f are the origin and $(-4,0)$. Next,

$$f'(x) = (x+4)^{1/3}+\tfrac13x(x+4)^{-2/3} = \tfrac43(x+4)^{-2/3}(x+3),$$

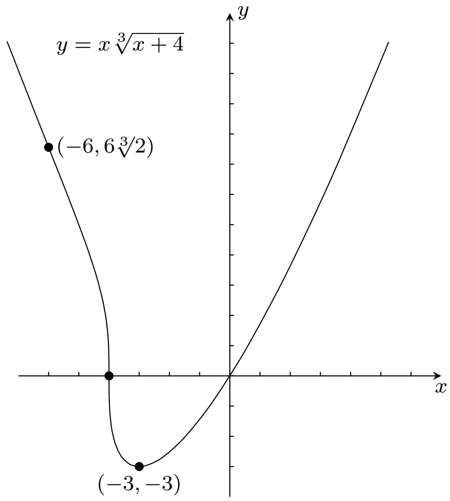
which is undefined if $x\neq-4$, positive if $x>-3$ and negative otherwise, so f is decreasing on $(-\infty,-3)$ and increasing on $(-3,\infty)$, with a local and global minimum value $f(-3) = -3$. Since

$$\lim_{t\to-4}\frac{f(t)-f(-4)}{t-(-4)} = \lim_{t\to-4}\frac{t}{(t+4)^{2/3}} = -\infty,$$

the graph of f has a vertical tangent line at $(-4,0)$.
Finally,

$$\begin{aligned} f''(x) &= -\tfrac{8}{9}(x+4)^{-5/3}(x+3) + \tfrac{4}{3}(x+4)^{-2/3} \\ &= \tfrac{4}{9}(x+4)^{-5/3}(x+6), \end{aligned}$$

so the graph of f is concave down on $(-6,-4)$, concave up on $(-\infty,-6)$ and $(4,\infty)$ with points of inflection at $(-6, 6\sqrt[3]{2})$ and $(-4,0)$. Follows a sketch of the graph of f.



13. The function f is continuous on $[-1,1]$ (since it is continuous on $\mathbb{R}$), so by the Extreme Value Theorem it attains global extreme values on $[-1,1]$. Now

$$f'(x) = 2\sqrt[3]{x} + \tfrac{1}{3}(2x-1)x^{-2/3} = \tfrac{1}{3}x^{-2/3}(8x-1)$$

is undefined at 0 and vanishes at $\frac{1}{8}$, each of which is in $(-1,1)$, and is otherwise defined and non-zero. Comparing

$$f(-1) = 3, \qquad f(0) = 0, \qquad f\left(\tfrac{1}{8}\right) = -\tfrac{3}{8} \qquad \text{and} \qquad f(1) = 1$$

reveals that the global minimum value of f on $[-1,1]$ is $-\frac{3}{8}$ and the global maximum value of f on $[-1,1]$ is 3.

14. If x denotes the distance (measured in kilometres) between P and the point directly across the river from A, then the cost of the pipeline is given by

$$C = 15\sqrt{x^2+1} + 9(3-x)$$

thousand dollars (the first term represents the cost of the part under the water and the second term represents the cost of the part along the riverside). Now

$$\frac{dC}{dx} = \frac{15x}{\sqrt{x^2+1}} - 9$$

is defined for all x, and is equal to zero if, and only if, $5x = 3\sqrt{x^2+1}$, *i.e.*, $16x^2 = 9$ and $x > 0$, or $x = \frac{3}{4}$. Since $\frac{dC}{dx} < 0$ if $x < \frac{3}{4}$, and $\frac{dC}{dx} > 0$ if $x > \frac{3}{4}$, it follows that the total cost of the pipeline is least if $x = \frac{3}{4}$; *i.e.*, if P is $\frac{9}{4}$ km from T.

15. a. Integrating term-by-term gives

$$\int \left(\frac{e^x}{4} + \frac{4^x}{e} + \frac{e}{4}\right) dx = \tfrac{1}{4}e^x + 4^x/(e\log 4) + ex/4 + C.$$

b. Expanding, simplifying and integrating term-by-term gives

$$\begin{aligned} \int (2+\sin x)\sec^2 x\,dx &= \int \left(2\sec^2 x + \sec x\tan x\right) dx \\ &= 2\tan x + \sec x + C. \end{aligned}$$

c. Integrating term-by-term gives

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (a\sin x + b\cos x)\,dx = (-a\cos x + b\sin x)\Big|_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = 2b.$$

16. Since

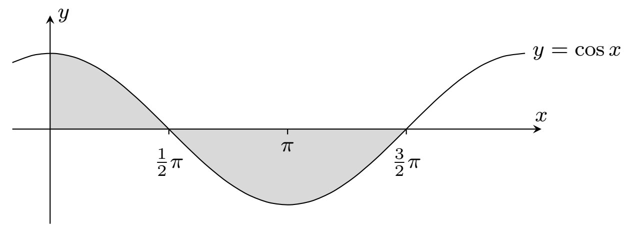
$$\int f'(x)\,dx = \int (2x - 3 + 4/x)\,dx = x^2 - 3x + 4\log|x| + C,$$

it follows that

$$f(x) = \begin{cases} x^2 - 3x + 4\log(-x) - 1 & \text{if } x < 0, \text{ and} \\ x^2 - 3x + 4\log x + 2 & \text{if } x > 0, \end{cases}$$

where the constants of integration are determined, respectively, by the conditions $f(-1) = 3$ and $f(1) = 0$.

17. The region in question is sketched and shaded below.



The area of the region is equal to

$$\int_0^{\frac{1}{2}\pi} \cos x\,dx + \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} (-\cos x)\,dx = (\sin x)\Big|_0^{\frac{1}{2}\pi} + (-\sin x)\Big|_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} = 3.$$

18. If $[0,2]$ is partitioned into n subintervals of equal length then $\Delta x_i = 2/n$ and $x_i = 2i/n$ for $i = 0,\dots,n$, and the terms of the corresponding right endpoint Riemann sum of $f(x) = xe^x$ are $f(x_i)\Delta x_i = \left(\frac{2i}{n}\right)e^{\frac{2i}{n}}\left(\frac{2}{n}\right)$, for $i = 1,\dots,n$. Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{2i}{n}\right)e^{\frac{2i}{n}}\left(\tfrac{2}{n}\right) = \int_0^2 xe^x\,dx.$$

To evaluate the limit first note that

$$\sum_{i=1}^{n} \left(\tfrac{2i}{n}\right)e^{\frac{2i}{n}}\left(\tfrac{2}{n}\right) = \frac{4}{n^2}\sum_{i=1}^{n} i\alpha^i,$$

where $\alpha = e^{2/n}$. A formula for the sum of a geometric progression gives

$$\alpha^i + \cdots + \alpha^n = \frac{\alpha^i(\alpha^{n-i+1}-1)}{\alpha-1} = \frac{\alpha^{n+1}-\alpha^i}{\alpha-1},$$

and therefore (summing another geometric progression on the second line)

$$\begin{aligned} \sum_{i=1}^{n} i\alpha^i &= \sum_{i=1}^{n}(\alpha^i + \cdots + \alpha^n) = \sum_{i=1}^{n} \frac{\alpha^{n+1}-\alpha^i}{\alpha-1} \\ &= \frac{n\alpha^{n+1}}{\alpha-1} - \frac{1}{\alpha-1}\sum_{i=1}^{n}\alpha^i = \frac{n\alpha^{n+1}}{\alpha-1} - \frac{\alpha(\alpha^n-1)}{(\alpha-1)^2}. \end{aligned}$$

Now the fact that $(e^t-1)/t \to 1$ as $t \to 0$ (plus limit laws) gives

$$\begin{aligned} \lim_{n\to\infty}\left\{\frac{4}{n^2}\cdot\frac{n\alpha^{n+1}}{\alpha-1}\right\} &= \lim_{n\to\infty} \frac{4e^{2+2/n}}{n(e^{2/n}-1)} \\ &= 2e^2 \lim_{n\to\infty} \frac{2/n}{e^{2/n}-1} \\ &= 2e^2, \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty}\left\{\frac{4}{n^2}\cdot\frac{\alpha(\alpha^n-1)}{(\alpha-1)^2}\right\} &= \lim_{n\to\infty} \frac{4e^{2/n}(e^2-1)}{n^2(e^{2/n}-1)^2} \\ &= (e^2-1)\lim_{n\to\infty}\left(\frac{2/n}{e^{2/n}-1}\right)^2 \\ &= e^2-1. \end{aligned}$$

Finally, combining these results yields

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{2i}{n}\right)e^{\frac{2i}{n}}\left(\tfrac{2}{n}\right) = 2e^2 - (e^2-1) = e^2+1.$$

19. a. $F(1) = \int_1^1 f(t)\,dt = 0,$ b. $F(3) = \int_1^9 f(t)\,dt = 4$ and

c. $F'(x) = 2xf(x^2)$ (by the first form of the Fundamental Theorem of Calculus and the Chain Rule).