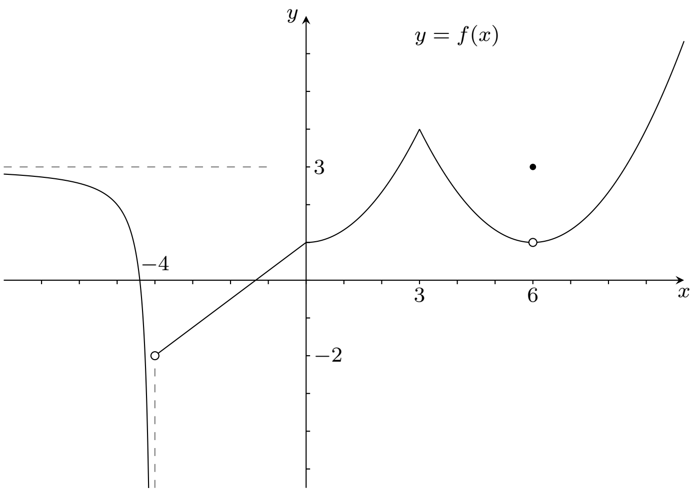
1. The graph of a function f is displayed below.



Determine each of the following. Where appropriate, use ∞, $-\infty$ or *undefined*.

a. $\lim_{x\to-\infty} f(x)$ b. $\lim_{x\to-4^-} f(x)$ c. $\lim_{x\to-4} f(x)$ d. $\lim_{x\to 6} f(x)$ e. $f(6)$

f. all numbers at which f is continuous but not differentiable

2. Evaluate each of the following. Use ∞, $-\infty$ or *undefined* as appropriate.

a. $\displaystyle\lim_{x\to-2} \frac{6-x-2x^2}{x^2-4}$ b. $\displaystyle\lim_{x\to 6} \frac{36-x^2}{\sqrt{x+3}-\sqrt{x^2-27}}$

c. $\displaystyle\lim_{x\to 0^-} \left\{\frac{1}{x}-\frac{2}{x^2}\right\}$ d. $\displaystyle\lim_{x\to-\infty} \left\{\frac{x^2-5x-6}{8x^3-27}-4\right\}$

3. Find and classify the discontinuities of the function f, defined by

$$f(x)=\begin{cases}\sqrt{3-x} & \text{if } x<-1,\\ \frac{1}{2}|2x+1| & \text{if } -1\leqslant x<1, \text{ and}\\ \dfrac{-3x}{x^2-x-2} & \text{if } x\geqslant 1 \text{ and } x\neq 2.\end{cases}$$

4. Suppose that the function f is differentiable on $[0,3]$. If $f'(x)\leqslant 2$ for all real values of x and $f(0)=4$, what is the largest possible value of $f(3)$?

5. Use the definition of the derivative to find $f'(x)$, where $f(x)=\dfrac{2}{3-x}$.

6. Find $\frac{dy}{dx}$ for each of the following.

a. $y=4x^3-\dfrac{2}{\sqrt[5]{x^3}}-\dfrac{5^{3x}}{2}+\cot x-\log_2 x+\frac{1}{2}e^3$ b. $y=\dfrac{\tan^2(e^x-3)}{3x^2+5}$

c. $y=(x^2+1)^{\csc x}$ d. $y=\log\dfrac{(6x-5)^2\sqrt[4]{x^3-2x+1}}{x^3(x^2-2x)^7}$

7. The position of a particle at time t is given by $s=t^3-3t^2$ for $t\geqslant 0$, where s is measured in metres and t is measured in seconds.

a. Find the velocity and acceleration functions of the particle.

b. When is the particle at rest? What is the acceleration of the particle when it is at rest?

c. Find the velocity of the particle after four seconds.

d. When is the acceleration of the particle zero?

8. Find the 25^{th} derivative of the function f, defined by $f(x)=x\sin 2x$.

9. At which values of x is the tangent line to the graph of $y=(x^2+2x-2)e^{2x}$ horizontal?

10. For the curve $\mathscr{C}$ defined by $x^3+y^3=9xy$, find:

a. $\dfrac{dy}{dx}$, and all points on $\mathscr{C}$ at which the tangent line has slope -1;

b. an equation of the normal line to $\mathscr{C}$ at the point $(2,4)$;

c. $\dfrac{d^2y}{dx^2}$, and simplify your answer.

11. Three metres above the ground a fly is flying horizontally at a rate of four metres per minute. It passes over a small rock at noon. How fast is the distance between the fly and the rock increasing one minute later?

12. Sketch the graph of the function f, defined by

$$f(x)=\frac{-1}{x^2-1}.$$

Be sure that your solution includes all asymptotes, intervals of monotonicity and concavity, and points of interest.

13. Find the absolute extrema of the function f defined by $f(x)=\frac{1}{8}x^3-\frac{3}{2}x$ on the closed interval $[-4,3]$.

14. A cylinder with a closed top (and bottom) must contain 16π cubic metres. What is the minimum amount of material (surface area) which can be used?

15. a. Use differentiation to verify that

$$\int \sec x\,dx=\log|\sec x+\tan x|+C.$$

b. Use the result of Part a to evaluate

$$\int_0^{\frac{1}{4}\pi}\sec x\,dx.$$

16. Evaluate each of the following integrals.

a. $\displaystyle\int_1^2\frac{(2x-1)^2}{4x}\,dx$ b. $\displaystyle\int\frac{x^2-9}{x-3}\,dx$ c. $\displaystyle\int\left\{\sqrt[5]{x}-\frac{5}{x}+5^x\right\}dx$

17. Given $f''(x)=6x-2\cos x$, $f'(0)=-1$ and $f(0)=4$, find $f(x)$.

18. Let $\mathscr{R}$ be the region bounded by the graph of $y=1+4/x$, the x axis and the lines defined by $x=1$ and $x=5$.

a. Sketch and shade $\mathscr{R}$.

b. Find the exact value of the area of $\mathscr{R}$.

c. Find an approximation of the area of $\mathscr{R}$ using a Riemann sum with four equal subintervals and right endpoints.

19. Evaluate $\displaystyle\frac{d}{dx}\int_x^7\sqrt{\tan\vartheta}\,d\vartheta$.

1. The graph of f reveals that a. $\lim_{x\to-\infty} f(x) = 3$, b. $\lim_{x\to-4^-} f(x) = -\infty$, c. $\lim_{x\to-4} f(x)$ is undefined (because the limit from the left is $-\infty$ and the limit from the right is -2), d. $\lim_{x\to 6} f(x) = 1$, e. $f(6) = 3$, and f. f is continuous but not differentiable at 0 and 3.

2. a. Since
$$\frac{6-x-2x^2}{x^2-4} = \frac{(3-2x)(2+x)}{(x-2)(x+2)} = \frac{3-2x}{x-2},$$
provided $x \neq -2$, one has
$$\lim_{x\to-2} \frac{6-x-2x^2}{x^2-4} = \lim_{x\to-2} \frac{3-2x}{x-2} = -\tfrac{7}{4},$$
by independence and direct substitution.

b. Rationalizing the denominator of the expression in the limit and then cancelling the common factor $6-x$ (using independence) gives
$$\begin{aligned}\lim_{x\to 6} &\frac{36-x^2}{\sqrt{x+3}-\sqrt{x^2-27}} \cdot \frac{\sqrt{x+3}+\sqrt{x^2-27}}{\sqrt{x+3}+\sqrt{x^2-27}} \\ &= \lim_{x\to 6} \frac{(36-x^2)(\sqrt{x+3}+\sqrt{x^2-27}\,)}{30+x-x^2} \\ &= \lim_{x\to 6} \frac{(6+x)(\sqrt{x+3}+\sqrt{x^2-27}\,)}{5+x} \\ &= \tfrac{72}{11},\end{aligned}$$
by direct substitution.

c. As $x \to 0^-$,
$$\frac{1}{x} \to -\infty \qquad \text{and} \qquad 1 - \frac{2}{x} \to \infty,$$
and therefore
$$\frac{1}{x} - \frac{2}{x^2} = \frac{1}{x}\left(1 - \frac{2}{x}\right) \to -\infty.$$

d. As $x \to -\infty$, $1/x \to 0$, and (by arithmetical properties of limits)
$$\frac{1+5/x-6/x^2}{8-27/x^3} \to \tfrac{1}{8},$$
and therefore
$$\lim_{x\to-\infty}\left\{\frac{x^2+5x-6}{8x^3-4} - 4\right\} = \lim_{x\to-\infty}\left\{\frac{1}{x}\cdot\frac{1+5/x-6/x^2}{8-27/x^3} - 4\right\} = -4.$$

3. Since the square root function is continuous on its domain, f is continuous at least on $(\,-\infty, -1\,)$. Now,
$$\lim_{x\to-1^-} f(x) = 2, \qquad \text{and} \qquad f(-1) = \lim_{x\to-1^+} f(x) = \tfrac{1}{2}$$
by direct substitution, so f has a jump discontinuity at -1. Since the absolute value function is everywhere continuous, f is continuous at least on $[\,-1, 1\,)$. Also, since
$$\lim_{x\to1^-} f(x) = \tfrac{3}{2}, \qquad \text{and} \qquad f(1) = \lim_{x\to1^+} f(x) = \tfrac{3}{2}$$
by direct substitution, f is continuous at 1. Finally,
$$\lim_{x\to2^\pm} f(x) = \lim_{x\to2^\pm} \frac{-3x}{(x-2)(x+1)} = \mp\infty,$$
so f has an infinite discontinuity at 2.

4. Since f is differentiable, and therefore continuous, on $[\,0,3\,]$, the Mean Value Theorem implies that $f(3) = f(0) + f'(\xi)(3-0) = 4 + 3f'(\xi)$, for some real number ξ in $(\,0,3\,)$. So $f(3) \leqslant 4 + 3\cdot 2 = 10$, since it is given that $f'(x) \leqslant 2$ for all real values of x. Therefore, the largest possible value of $f(3)$ is 10.

5. Since
$$f(t) - f(x) = \frac{2}{3-t} - \frac{2}{3-x} = \frac{2(t-x)}{(3-t)(3-x)},$$
provided neither x nor t is equal to 3, it follows that
$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = \lim_{t\to x} \frac{2}{(3-t)(3-x)} = \frac{2}{(3-x)^2},$$
by independence and direct substitution.

6. a. Using linearity, basic derivative formulæ, and the Chain Rule, one has
$$\frac{dy}{dx} = 12x^2 + \tfrac{6}{5}x^{-8/5} - \tfrac{3}{2}(\log 5)5^{3x} - \csc^2 x - \frac{1}{x\log 2}.$$

b. Using linearity, basic derivative formulæ, the Quotient Rule and the Chain Rule, one has
$$\begin{aligned}\frac{dy}{dx} &= \frac{\big(2e^x\tan(e^x-3)\sec^2(e^x-3)\big)(3x^2+5) - \big(\tan^2(e^x-3)\big)(6x)}{(3x^2+5)^2} \\ &= \frac{2\big(e^x(3x^2+5)\sec^2(e^x-3) - 3x\tan(e^x-3)\big)\tan(e^x-3)}{(3x^2+5)^2}.\end{aligned}$$

c. Logarithmic differentiation (together with basic derivative formulæ and the Product and Chain Rules) gives
$$\begin{aligned}\frac{dy}{dx} &= y\frac{d}{dx}\{\log y\} = (x^2+1)^{\csc x}\left(\frac{2x\csc x}{x^2+1} - (\csc x\cot x)\log(x^2+1)\right) \\ &= (x^2+1)^{\csc x - 1}(\csc x)\big(2x - (\cot x)(x^2+1)\log(x^2+1)\big).\end{aligned}$$

d. Since
$$y = 2\log(6x-5) + \tfrac{1}{4}\log(x^3-2x+1) - 10\log x - 7\log(x-2)$$
(at least for $x > 2$), basic derivative formulæ and the Chain Rule yield
$$\frac{dy}{dx} = \frac{12}{6x-5} + \frac{3x-2}{4(x^3-2x+1)} - \frac{10}{x} - \frac{7}{x-2}.$$

7. a. If the position of the particle is given by $s = t^3 - 3t^2$ for $t \geqslant 0$ then the velocity and acceleration are given for $t \geqslant 0$, respectively, by
$$v = \frac{ds}{dt} = 3t^2 - 6t = 3t(t-2) \qquad \text{and} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 6(t-1).$$

b. The particle is a rest when its velocity is zero, *i.e.*, when $t = 0$ or when $t = 2$. Initially ($t = 0$) the particle's acceleration is $-6\,\mathrm{m/s^2}$, and after two seconds ($t = 2$) the particle's acceleration is $6\,\mathrm{m/s^2}$.

c. The velocity of the particle after four seconds is $24\,\mathrm{m/s}$.

d. The acceleration of the particle is zero after one second.

8. Computing the first few derivatives of f and keeping track of the pattern,
$$\begin{aligned}f(x) &= x\sin 2x, \\ f'(x) &= \sin 2x + 2x\cos 2x, \\ f''(x) &= 2^1(2\cos 2x - 2x\sin 2x), \\ f^{(3)}(x) &= -2^2(3\sin 2x + 2x\cos 2x), \\ f^{(4)}(x) &= -2^3(4\cos 2x - 2x\sin 2x), \\ f^{(5)}(x) &= 2^4(5\sin 2x + 2x\cos 2x), \qquad \text{\&c.},\end{aligned}$$
reveals that $f^{(25)}(x) = 2^{24}(25\sin 2x + 2x\cos 2x)$.

9. If $y = (x^2+2x-2)e^{2x}$ then
$$\frac{dy}{dx} = 2(x+1)e^{2x} + 2(x^2+2x-2)e^{2x} = 2(x^2+3x-1)e^{2x}.$$
The tangent line to the graph of $y = (x^2+2x-2)e^{2x}$ is horizontal if, and only if, $\frac{dy}{dx} = 0$, *i.e.*, $x^2 + 3x = 1$ (since the exponential factor is never zero), or $\big(x + \frac{3}{2}\big)^2 = \frac{13}{4}$, which gives $x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{13}$.

10. a. By the Chain Rule one has
$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}, \qquad \text{or} \qquad \frac{dy}{dx} = \frac{x^2-3y}{3x-y^2},$$
where the given equation defines y as an implicit function of x. The tangent line to $\mathscr{C}$ has slope -1 if, and only if, $\frac{dy}{dx} = -1$, *i.e.*, $x^2 - 3y = y^2 - 3x$, or $(x-y)(x+y+3) = 0$, which gives $x = y$ or $x + y = -3$. Where $x = y$ on $\mathscr{C}$, $2x^3 = 9x^2$, or $x^2(2x-9) = 0$, which gives $x = \frac{9}{2}$ (and therefore $y = \frac{9}{2}$ also) since the only y-intercept of $\mathscr{C}$ is the origin, at which $\frac{dy}{dx}$ is undefined (in fact, the equation of $\mathscr{C}$ does not define y as an implicit function of x on an open interval containing the origin). If $x + y = -3$ on $\mathscr{C}$ then, since $x^3 + y^3 = (x+y)(x^2-xy+y^2)$, it follows that $-3(x^2-xy+y^2) = 9xy$, or $0 = x^2 + 2xy + y^2 = (x+y)^2 = (-3)^2 = 9$, which is impossible, so there are no points on $\mathscr{C}$ for which $x + y = -3$. Therefore, the only point on $\mathscr{C}$ at which the tangent line has slope -1 is $\big(\frac{9}{2}, \frac{9}{2}\big)$.

b. The slope of the tangent line to $\mathscr{C}$ at $(2,4)$ is

$$\left.\frac{dy}{dx}\right|_{\substack{x=2,\\ y=4}} = \frac{2^2 - 3\cdot 4}{3\cdot 2 - 4^2} = \tfrac{4}{5},$$

so the slope of the normal line, since it is orthogonal to the tangent line, is $-\frac{5}{4}$. Therefore, $y - 4 = -\frac{5}{4}(x-2)$, or $5x + 4y = 26$, is an equation of the normal line to $\mathscr{C}$ at the point $(2,4)$.

c. Applying the Quotient Rule and the Chain Rule (where y is an implicit function of x), and then replacing $\frac{dy}{dx}$ by its value compute in Part a gives gives

$$\frac{d^2y}{dx^2} = \frac{\big(2x - 3\frac{dy}{dx}\big)(3x - y^2) - (x^2 - 3y)\big(3 - 2y\frac{dy}{dx}\big)}{(3x-y^2)^2}.$$

Now

$$\begin{aligned}\big(2x - 3\tfrac{dy}{dx}\big)(3x - y^2) &= \frac{\big(2x(3x-y^2) - 3(x^2-3y)\big)(3x-y^2)}{3x-y^2}\\ &= \frac{(3x^2 - 2xy^2 + 9y)(3x-y^2)}{3x-y^2}\\ &= \frac{9x^3 - 9x^2y^2 + 2xy^4 + 27xy - 9y^3}{3x-y^2},\end{aligned}$$

and

$$\begin{aligned}(x^2 - 3y)\big(3 - 2y\tfrac{dy}{dx}\big) &= \frac{(x^2-3y)\big(3(3x-y^2) - 2y(x^2-3y)\big)}{3x-y^2}\\ &= \frac{(x^2-3y)(-2x^2y + 9x + 3y^2)}{3x-y^2}\\ &= \frac{-2x^4y + 9x^3 + 9x^2y^2 - 27xy - 9y^3}{3x-y^2},\end{aligned}$$

and so the numerator of their difference is

$$2x^4y - 18x^2y^2 + 2xy^4 + 54xy = 2xy(x^3 - 9xy + y^3 + 27) = 54xy,$$

since $x^3 + y^3 = 9xy$ on $\mathscr{C}$. Combining this with the first calculation give

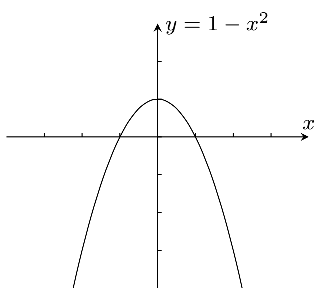
$$\frac{d^2y}{dx^2} = \frac{54xy}{(3x-y^2)^3}.$$

11. When the fly is x metres past the rock its distance s from the rock (measured in metres) satisfies $s^2 = x^2 + 9$, and therefore

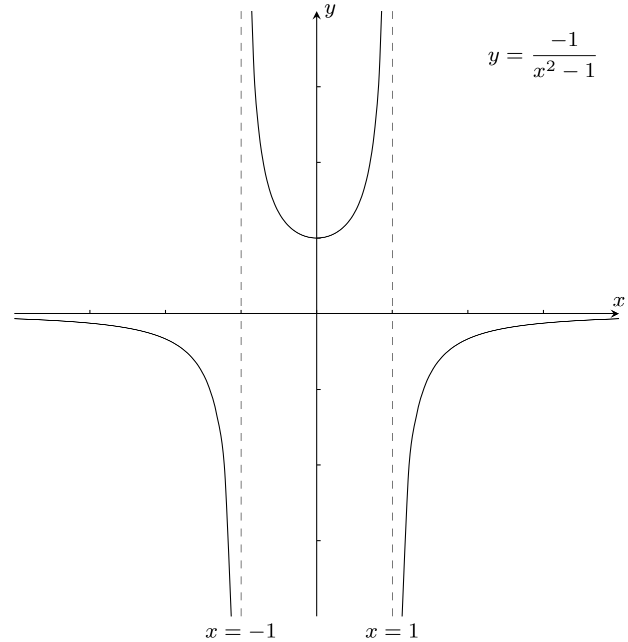
$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}, \qquad \text{or} \qquad \frac{ds}{dt} = \frac{4x}{s},$$

since it is given that $\frac{dx}{dt} = 4$ At one minute past noon $x = 4$, so $s = 5$, and hence $\frac{ds}{dt} = \frac{16}{5}$. Therefore, at one minute past noon, the distance between the fly and the rock is increasing at a rate of $\frac{16}{5}$ metres per minute.

12. The given function is the reciprocal of the quadradic function defined by $g(x) = 1 - x^2$, so calculus is unnecessary. The graph of g is displayed below, with unit lengths marked alone the coordinate axes.



Since taking reciprocals reverses order and preserves sign, it follows that the graph of f is as follows (with unit lengths marked along the coordinate axes).



The function f is increasing on $[\,0,1\,)$ and $(\,1,\infty\,)$, and decreasing on $(\,-\infty,-1\,)$ and $(\,-1,0\,]$, with no global extrema and a local minimum at $(0,1)$; it is concave up on $(\,-1,1\,)$ and concave down on $(\,-\infty,-1\,)$ and $(\,1,\infty\,)$, with no points of inflection. The graph of f has no x intercepts and its only y intercept is $(0,1)$. The x-axis is the horizontal asymptote of the graph of f, and the lines defined by $x = -1$, $x = 1$ are the vertical asymptotes of the graph of f.

13. The given function is differentiable (and therefore also continuous) on the given interval, so its extreme values occur at the endpoints -4, 3, or where its derivative vanishes in $(\,-4,3\,)$. Now

$$f'(x) = \tfrac{3}{8}x^2 - \tfrac{3}{2} = \tfrac{3}{2}(x^2 - 4)$$

vanishes if $x = \pm 2$, each of which belong to the interval $(\,-4,3\,)$. Comparing

$$f(-4) = -2, \qquad f(-2) = 2, \qquad f(2) = -2 \qquad \text{and} \qquad f(3) = -\tfrac{9}{8},$$

reveals that the maximum value of f on $[\,-4,3\,]$ is 2 (which occurs at the critical number -2) and the minimum value of f on $[\,-4,3\,]$ is -2 (which occurs at the endpoint -4 and at the cricital number 2).

14. If r denotes the radius of the cylinder and h denotes the height of the cylinder (each measured in metres) then its surface area (an idealization of the amount of material needed to construct it) is given by $S = 2\pi r^2 + 2\pi rh = 2\pi(r^2 + rh)$. Since the volume of the cylinder satisfies $16\pi = \pi r^2 h$, it follows that $h = 16r^{-2}$, and therefore $S = 2\pi(r^2 + 16r^{-1})$, for $r > 0$. Then

$$\frac{dS}{dr} = 2\pi(2r - 16r^{-2}) = 4\pi r^{-2}(r^3 - 8), \quad \text{and} \quad \frac{d^2S}{dr^2} = 4\pi(1 + 16r^{-3}),$$

which implies (since the second derivative is positive if r is) that the minimum value of S on $(\,0,\infty\,)$ occurs where $\frac{dS}{dr} = 0$, *i.e.*, where $r = 2$. Therefore, the least possible surface area of a cylinder as required is 24π square metres (which is obtained by constructing a cylinder of radius 2 metres and height 4 metres).

15. a. Since

$$\begin{aligned}\frac{d}{dx}\big\{\log|\sec x + \tan x|\big\} &= \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \frac{(\sec x)(\tan x + \sec x)}{\sec x + \tan x}\\ &= \sec x,\end{aligned}$$

it follows that

$$\int \sec x\,dx = \log|\sec x + \tan x| + C.$$

b. The result of Part a implies that

$$\int_0^{\frac{1}{4}\pi} \sec x\,dx = \log|\sec x + \tan x|\,\Big|_0^{\frac{1}{4}\pi} = \log(1 + \sqrt{2}).$$

16. a. Expanding, and then dividing and integrating termwise yields

$$\int_1^2 \frac{(2x-1)^2}{4x}\,dx = \int_1^2 \frac{4x^2-4x+1}{4x}\,dx = \int_1^2 \bigl\{x - 1 + \tfrac{1}{4}x^{-1}\bigr\}\,dx$$
$$= \bigl\{\tfrac{1}{2}x^2 - x + \tfrac{1}{4}\log x\bigr\}\Big|_1^2$$
$$= \tfrac{1}{2} + \tfrac{1}{4}\log 2.$$

b. Cancelling the common factor $x - 3$ from the numerator and denominator of the integrand and then integrating termwise gives

$$\int \frac{x^2-9}{x-3}\,dx = \int (x+3)\,dx = \tfrac{1}{2}x^2 + 3x + C.$$

c. Termwise integration yields

$$\int \left\{\sqrt[5]{x} - \frac{5}{x} + 5^x\right\} dx = \tfrac{5}{6}x^{6/5} - 5\log|x| + \frac{5^x}{\log 5} + C.$$

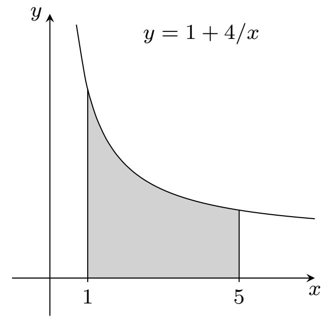
17. By the Fundamental Theorem of Calculus,

$$f'(x) = -1 + \int_0^x (6t - 2\cos t)\,dt = -1 + 3x^2 - 2\sin x,$$

and

$$f(x) = 4 + \int_0^x (-1 + 3t^2 - 2\sin t)\,dt = 2 - x + x^3 + 2\cos x.$$

18. a. A sketch of $\mathscr{R}$ is displayed below.



b. The area of $\mathscr{R}$ is equal to

$$\int_1^5 (1 + 4/x)\,dx = (x + 4\log x)\Big|_1^5 = 4 + 4\log 5.$$

c. If $[1,5]$ is divided into four subintervals of equal length then $\Delta x_i = 1$ for $i = 1,2,3,4$ and $x_i = 1 + i$, for $i = 0,1,2,3,4$. The corresponding right endpoint Riemann sum is

$$\sum_{i=1}^4 f(x_i)\Delta x_i = (1 + 4/2) + (1 + 4/3) + (1 + 4/4) + (1 + 4/5)$$
$$= \tfrac{137}{15}.$$

19. One has

$$\frac{d}{dx}\int_x^7 \sqrt{\tan\vartheta}\,d\vartheta = -\frac{d}{dx}\int_7^x \sqrt{\tan\vartheta}\,d\vartheta = -\sqrt{\tan x}$$

by the (first form of the) Fundamental Theorem of Calculus.