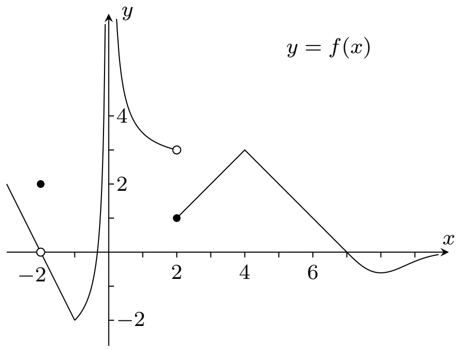
1. Given the graph of f below, evaluate each of the following. Use ∞, $-\infty$ or "does not exist" where appropriate.



a. $\lim_{x\to -2} f(x)$

b. $\lim_{x\to 0} f(x)$

c. $f'(4)$

d. $\lim_{x\to\infty} f(x)$

e. $\lim_{h\to 0} \dfrac{f(6+h)-f(6)}{h}$

f. $\lim_{x\to 2} [f(x)-2]^2$

2. Evaluate each of the following limits.

a. $\lim_{x\to 5} \dfrac{50-2x^2}{2x^2-9x-5}$ b. $\lim_{x\to -\infty} \dfrac{\sqrt{4x^6-9x}}{x^3}$ c. $\lim_{x\to\infty} (e^x - e^{2x})$

d. $\lim_{x\to 3^+} \dfrac{|6-2x|}{\sqrt{x-3}}$ e. $\lim_{x\to 0} \dfrac{6x}{\sin 3x\,\cos 4x}$

3. Find and classify the discontinuities of the function f defined by

$$f(x) = \begin{cases} \dfrac{x^2-4}{x^2-x-6} & \text{if } x \leqslant -1 \text{ and } x \neq -2, \\ \frac{1}{4}x + 1 & \text{if } -1 < x < 5, \text{ and} \\ \dfrac{1}{x^2-10x-24} & \text{if } x \geqslant 5 \text{ and } x \neq 12. \end{cases}$$

4. Use the limit definition of the derivative to find $f'(x)$, where $f(x) = \dfrac{1}{x^2+1}$.

5. Find $\frac{dy}{dx}$ for each of the following.

a. $y = 5^{\cot x} + \sec(4x^2) - 2e^{\pi+1}$ b. $y = \tan^3(xe^x)$

c. $y = \sqrt{\dfrac{x^3 \sin 2x}{(x+1)^5}}$ d. $e^{xy} - 3x^2 - 3y^2 = 2$ e. $y = \left(\dfrac{2x-3}{\cos x}\right)^x$

6. Consider the curve defined by $xy^2 - x^3 y = 6$.

a. Prove that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

b. Find all points on the curve whose x-coordinate is 1, and write an equation of the tangent line at each of these points.

7. Consider the function defined by $f(x) = x^3 - 7x - 10$.

a. Show that f has a zero in the interval $[-1, 4]$.

b. Find all numbers in the interval $(-1, 4)$ which satisfy the conclusion of the Mean Value Theorem.

c. Use Rolle's Theorem to show that there is a number c in $(-1, 3)$ such that $f'(c) = 0$.

8. A conical tank with its vertex down has a diameter of 8 m and a depth of 16 m. Water flows into the tank at a rate of 5 cubic metres per minute. Find the rate at which the water level is rising when the water is 10 m deep.

9. Find the absolute extrema on the interval $[1, e^4]$ of the function f defined by

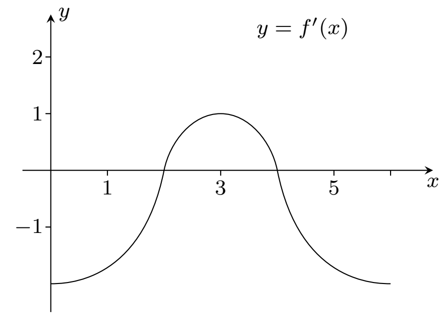
$$f(x) = \frac{\log x}{\sqrt{x}}.$$

10. Sketch the graph of the function f defined by

$$f(x) = \frac{(2x+3)(x-3)^2}{x^3} = 2 - \frac{9}{x} + \frac{27}{x^3}.$$

Include all features and points of interest in your solution.

11. The graph below is of a function f' on $[0, 6]$.



a. Give the interval(s) on which f is decreasing.

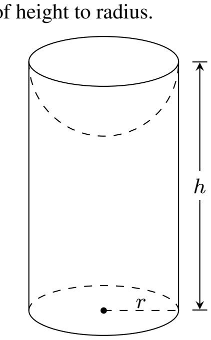
b. Give the interval(s) on which the graph of f is concave up.

c. Give the value of x at which f has a local maximum.

d. Give the value of x at which f assumes its global maximum value on $[0, 6]$.

e. Give the x-coordinate(s) of all point(s) of inflection of the graph of f.

12. A closed cylindrical tank with a flat bottom and an inverted hemispherical top is to have a volume of 13π. Find the dimensions that will minimize the cost of the metal to make the tank, and state the resulting ratio h/r of height to radius.



13. Find $x(t)$ if $x''(t) = e^t - 3\cos t + 6t$, $x'(0) = 3$ and $x(0) = 1$.

14. Evaluate each of the following integrals.

a. $\displaystyle\int (6e^x - \sqrt[3]{x^7} + \pi^5)\,dx$ b. $\displaystyle\int \frac{(x-1)^2}{x^3}\,dx$

c. $\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin^2 x + \cos x}{\sin^2 x}\,dx$ d. $\displaystyle\int_0^5 |x^2 - 9|\,dx$

15. Find the derivative with respect to x of $y = \displaystyle\int_{\sqrt{x}}^{1} \frac{t}{t^2+1}\,dt$.

16. Express

$$\int_1^3 (x^2 - x + 3)\,dx$$

as a limit of Riemann sums, and use summation formulæ and basic properties of limits to evaluate the integral.

No marks if you use the Fundamental Theorem of Calculus to evaluate the integral.

17. Evaluate the integral

$$\int_{-2}^{1} \sqrt{4 - x^2}\,dx$$

by interpreting it in terms of area.

18. Suppose that $\lim_{x\to\infty} f'(x) = 0$.

a. Must the graph of f have a horizontal asymptote? Justify your answer.

b. If f is a rational function, must the graph of f have a horizontal asymptote? Justify your answer.

1. Inspecting the graph of f gives
a. $\lim_{x\to -2} f(x) = 0$, b. $\lim_{x\to 0} f(x) = \infty$, c. $f'(4)$ is undefined,

d. $\lim_{x\to\infty} f(x) = 0$, e. $\lim_{h\to 0} \dfrac{f(6+h)-f(6)}{h} = -1$, f. $\lim_{x\to 2}[f(x)-2]^2 = 1$.

2. a. Since
$$\frac{50-2x^2}{2x^2-9x-5} = \frac{2(5-x)(5+x)}{(x-5)(2x+1)} = \frac{-2(5+x)}{2x+1}$$
for $x \neq 5$,
$$\lim_{x\to 5} \frac{50-2x^2}{2x^2-9x-5} = \lim_{x\to 5} \frac{-2(5+x)}{2x+1} = -\tfrac{20}{11}$$
by independence and direct substitution.
b. Since $x^3 = -\sqrt{x^6}$ for $x<0$, and $9/x^5 \to 0$ as $x \to -\infty$,
$$\lim_{x\to-\infty} \frac{\sqrt{4x^6-9x}}{x^3} = -\lim_{x\to-\infty}\sqrt{4-9/x^5} = -2.$$
c. Since $e^{2x}\to\infty$ and $e^{-x}\to 0$ ad $x\to\infty$, it follows that
$$\lim_{x\to\infty}(e^x - e^{2x}) = \lim_{x\to\infty}\big(e^{2x}(e^{-x}-1)\big) = -\infty.$$
d. Since $|3-x| = x-3$ if $x>3$, it follows that
$$\lim_{x\to 3^+}\frac{|6-2x|}{\sqrt{x-3}} = \lim_{x\to 3^+}\frac{2(x-3)}{\sqrt{x-3}} = 2\lim_{x\to 3^+}\sqrt{x-3} = 0,$$
by independence and direct substitution.
e. Since $(\sin t)/t \to 1$ as $t\to 0$, direct substitution and arithmetical limit laws imply that
$$\lim_{x\to 0}\frac{6x}{\sin 3x\,\cos 4x} = 2\lim_{x\to 0}\left\{\left(\frac{\sin 3x}{3x}\right)^{-1}\frac{1}{\cos 4x}\right\} = 2.$$

3. Since a rational function is continuous on every interval contained in its domain, f is continuous at least on $(-\infty,-2)$, $(-2,-1)$, $(-1,5)$, $(5,12)$ and $(12,\infty)$, and certainly discontinuous at -2 and 12 (neither of which belongs to the domain of f). It remains to determine the nature of the discontinuities of f at -2 and at 12, and to investigate the continuity of f at -1 and at 5. Now
$$\lim_{x\to-2} f(x) = \lim_{x\to-2}\frac{(x-2)(x+2)}{(x-3)(x+2)} = \lim_{x\to-2}\frac{x-2}{x-3} = \tfrac{4}{5}$$
by independence and direct substitution, f has a removable discontinuity at -2, and since $x^2-10x-24 = (x+2)(x-12) \to 0^{\pm}$ as $x\to 12^{\pm}$,
$$\lim_{x\to 12^{\pm}} f(x) = \lim_{x\to 12^{\pm}}\frac{1}{x^2-10x-24} = \pm\infty,$$
so f has an infinite discontinuity at 12. Next, from
$$f(-1) = \lim_{x\to -1^-} f(x) = \tfrac{3}{4} \qquad\text{and}\qquad \lim_{x\to-1^+} f(x) = \tfrac{3}{4},$$
it follows that f is continuous at -1. Finally,
$$\lim_{x\to 5^-} f(x) = \tfrac{9}{4} \qquad\text{and}\qquad f(5) = \lim_{x\to 5^+} f(x) = -\tfrac{1}{49}$$
by direct substitution, so f has a jump discontinuity at 5 (where f is continuous from the right).

4. Since
$$f(t)-f(x) = \frac{1}{t^2+1} - \frac{1}{x^2+1} = \frac{x^2-t^2}{(t^2+1)(x^2+1)} = \frac{(x-t)(x+t)}{(t^2+1)(x^2+1)},$$
it follows from the definition of the derivative of a function, that
$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{t\to x}\frac{-(x+t)}{(t^2+1)(x^2+1)} = \frac{-2x}{(x^2+1)^2}$$
by independence and direct substitution.

5. a. $\dfrac{dy}{dx} = -(\log 5)5^{\cot x}\csc^2 x + 8x\sec(4x^2)\tan(4x^2)$
b. $\dfrac{dy}{dx} = 3e^x(x+1)\tan^2(xe^x)\sec^2(xe^x)$
c. Logarithmic differentiation gives
$$\frac{dy}{dx} = y\frac{d}{dx}\log|y| = \tfrac{1}{2}y\frac{d}{dx}\Big\{3\log|x| + \log|\sin 2x| - 5\log|x+1|\Big\}$$
$$= \tfrac{1}{2}\sqrt{\frac{x^3\sin 2x}{(x+1)^5}}\left\{\frac{3}{x} + 2\cot 2x - \frac{5}{x+1}\right\}.$$
d. Differentiating implicitly with respect to x gives
$$ye^{xy} + xe^{xy}\frac{dy}{dx} - 6x - 6y\frac{dy}{dx} = 0, \qquad\text{or}\qquad \frac{dy}{dx} = \frac{6x-ye^{xy}}{xe^{xy}-6y}.$$
e. Logarithmic differentiation gives
$$\frac{dy}{dx} = y\frac{d}{dx}\log|y| = y\frac{d}{dx}\Big\{x\big(\log|2x-3| - \log|\cos x|\big)\Big\}$$
$$= \left(\frac{2x-3}{\cos x}\right)^x\left\{\log\frac{2x-3}{\cos x} + \frac{2x}{2x-3} + x\tan x\right\}$$

6. a. Differentiating the relation $xy^2 - x^3y = 6$ implicitly with respect to x gives
$$y^2 + 2xy\frac{dy}{dx} - 3x^2y - x^3\frac{dy}{dx}, \qquad\text{or}\qquad \frac{dy}{dx} = \frac{3x^2y-y^2}{2xy-x^3},$$
as required.
b. If $xy^2 - x^3y = 6$ and $x=1$ then $y^2 - y = 6$, or $(y+2)(y-3) = 0$, and so $y=-2$ or $y=3$. Now,
$$\left.\frac{dy}{dx}\right|_{\substack{x=1,\\y=-2}} = 2, \qquad\text{and so}\qquad y+2 = 2(x-1), \quad\text{or}\quad 2x-y = 4,$$
is an equation of the line tangent to the given curve at the point $(1,-2)$. Also,
$$\left.\frac{dy}{dx}\right|_{\substack{x=1,\\y=3}} = 0, \qquad\text{and so}\qquad y = 3$$
is an equation of the line tangent to the given curve at the point $(1,3)$.

7. a. Since f is a polynomial function, f is continuous on $\mathbb{R}$ and therefore on $[-1,4]$. Now $f(-1) = -4 < 0 < 26 = f(4)$, so the Intermediate Value Theorem implies that there is a real number r in $(-1,4)$ such that $f(r) = 0$, as required.
b. Since f is a polynomial function, it satisfies the hypotheses of the Mean Value Theorem on the interval $[-1,4]$. The equation $f(4)-f(-1) = f'(\xi)\big(4-(-1)\big)$ is equivalent to $30 = 5(3\xi^2-7)$, or $\xi^2 = \frac{13}{3}$, and so $\frac{1}{3}\sqrt{39}$ is the only number in $(-1,4)$ which satisfies the conclusion of the Mean Value Theorem in this case.
c. Since f is a polynomial function, it satisfies the hypotheses of Rolle's Theorem on the interval $[-1,4]$. Now $f(-1) = -4 = f(3)$, so Rolle's Theorem implies that there is a real number c in $(-1,3)$ such that $f'(c) = 0$.

8. The volume of water in the tank is $V = \frac{1}{3}\pi r^2 h$ (cubic metres), where h is the depth of the water, r is the radius of its upper surface, and $h = 4r$ by similarity. By the Chain Rule,
$$5 = \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = \pi r^2\frac{dh}{dt}.$$
When the water is 10 m deep, $\pi r^2 = \frac{25}{4}\pi\,\mathrm{m}^2$, and so $\frac{dh}{dt} = 4/(5\pi)$ m/min. Therefore, the water level in the tank is rising at a rate of $4/(5\pi)$ metres per minute when the water is 10 metres deep.

9. Since
$$f'(x) = \frac{1}{x\sqrt{x}} - \frac{\log x}{2x\sqrt{x}} = \frac{2-\log x}{2x\sqrt{x}},$$
$f'(x) = 0$ if, and only if, $\log x = 2$, or $x = e^2$, so e^2 is the critical number of f. Now $f(1) = 0$, $0 < f(2) = 2/e < 1$ and $0 < f(4) = 4/e^2 = (2/e)^2 < 2/e$, so the largest value of f on $[1,e^4]$ is $f(e^2) = 2/e$ and the smallest value of f on $[1,e^4]$ is $f(1) = 0$.

10. The domain of f is $\mathbb{R}\setminus\{0\}$, and $f(x) \to \pm\infty$ as $x \to 0^{\pm}$, so the y-axis is the vertical asymptote of the graph of f (and f has no global extrema). As $x \to \pm\infty$, $f(x) \to 2$, so the line defined by $y = 2$ is the horizontal asymptote of the graph of f. The axis intercepts of the graph of f are $(-\frac{3}{2}, 0)$ and $(3, 0)$. Since

$$f'(x) = \frac{9}{x^2} - \frac{81}{x^4} = \frac{9(x^2-9)}{x^4}$$

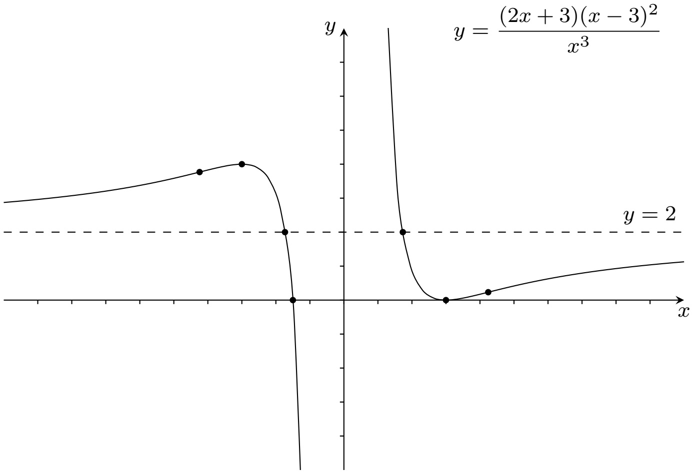
is positive if $|x| > 3$ and negative if $|x| < 3$ and $x \neq 0$, f is increasing on $(-\infty, -3)$ and on $(3, \infty)$, and decreasing on $(-3, 0)$ and on $(0, 3)$, with a local maximum at $(-3, 4)$ and a local minimum at $(3, 0)$. Next, since

$$f''(x) = 9\left\{-\frac{2}{x^3} + \frac{36}{x^5}\right\} = \frac{18(18-x^2)}{x^5}$$

is positive if $x < -3\sqrt{2}$ or $0 < x < 3\sqrt{2}$ and negative if $-3\sqrt{2} < x < 0$ or $x > 3\sqrt{2}$, it follows that the graph of f is concave up on $(-\infty, -3\sqrt{2})$ and on $(0, 3\sqrt{2})$, and concave down on $(-3\sqrt{2}, 0)$ and on $(3\sqrt{2}, \infty)$, with points of inflection at $(\pm 3\sqrt{2}, 2 \mp \frac{5}{4}\sqrt{2})$. The graph of f meets its horizontal asymptote where

$$0 = -\frac{9}{x} + \frac{27}{x^3} = \frac{9(3-x^2)}{x^3},$$

i.e., at $(\pm\sqrt{3}, 2)$. Below is a sketch of the graph of f, with unit lengths marked along the coordinate axes, the horizontal asymptote drawn as a dashed line and the points of interest emphasized.



11. a. Since the function f is differentiable and therefore continuous on $[0, 6]$, and since $f'(x)$ is negative if $0 < x < 2$ or $4 < x < 6$, it follows that f is decreasing on $[0, 2]$ and on $[4, 6]$.

b. Since f' is increasing on $[0, 3]$, the graph of f is concave up on $[0, 3]$.

c. Since f is continuous on $[0, 6]$, $f' > 0$ on $(2, 4)$, $f'(4) = 0$ and $f' < 0$ on $(0, 2)$ and on $(4, 6)$, it follows that f has a local maximum at 4 and nowhere else in $(0, 6)$.

d. Since f is continuous on $[0, 6]$, its maximum value occurs at (at least) one of 0, 2, 4, 6 (the first and last of which are endpoints and the second and third of which are critical numbers). Next, observe that

$$\int_0^2 f'(x)\,dx < -2, \qquad \int_2^4 f'(x)\,dx < 2 \qquad \text{and} \qquad \int_4^6 f'(x)\,dx < -2,$$

because each of the first and third integrals is the opposite of the area of a region which contains a triangle of base 2 and height 2, and the second integral is the area of a region which is contained in a rectangle of base 2 and height 2. Since, by the Fundamental Theorem of Calculus,

$$f(x) = f(0) + \int_0^x f'(t)\,dt,$$

the displayed inequalities imply that $f(0)$ is greater than $f(2)$, $f(4)$ and $f(6)$; therefore, the largest value of f on $[0, 6]$ occurs at 0.

e. Since f' is increasing on $[0, 3]$ and decreasing on $[3, 6]$, the graph of f changes concavity and therefore has a point of inflection at 3.

12. The volume of the can is given by

$$13\pi = \pi r^2 h - \tfrac{2}{3}\pi r^3, \qquad \text{and so} \qquad h = 13r^{-2} + \tfrac{2}{3}r.$$

The cost of the tank (if all of its parts are made from equally costly material) is proportional to its surface area, S, which is given by

$$S = 3\pi r^2 + 2\pi r h = 3\pi r^2 + 2\pi r(13r^{-2} + \tfrac{2}{3}r) = \tfrac{13}{3}\pi(r^2 + 6r^{-1})$$

for $r > 0$. Now

$$\frac{dS}{dr} = \tfrac{13}{3}\pi(2r - 6r^{-2}) = \tfrac{26}{3}\pi r^{-2}(r^3 - 3)$$

has the unique critical number $\sqrt[3]{3}$ on $(0, \infty)$, on which

$$\frac{d^2S}{dr^2} = \tfrac{26}{3}\pi(1 + 6r^{-3})$$

is positive. By the Second Derivative Test for global extrema, the least expensive can has radius $\sqrt[3]{3}$ and height $\frac{13}{3}\sqrt[3]{3} + \frac{2}{3}\sqrt[3]{3} = 5\sqrt[3]{3}$, so the ratio of height to radius is 5.

13. If $x''(t) = e^t - 3\cos t + 6t$ and $x'(0) = 3$ then

$$x'(t) = 3 + \int_0^t (e^\tau - 3\cos\tau + 6\tau)\,d\tau = e^t - 3\sin t + 3t^2 + 2,$$

and if in addition $x(0) = 1$ then

$$x(t) = 1 + \int_0^t (e^\tau - 3\sin\tau + 3\tau^2 + 2)\,d\tau = e^t + 3\cos t + t^3 + 2t - 3.$$

14. a. Integrating termwise gives

$$\int (6e^x - x^{7/3} + \pi^5)\,dx = 6e^x - \tfrac{3}{10}x^{10/3} + \pi^5 x + C.$$

b. Expanding the integrand and integrating termwise gives

$$\begin{aligned}\int \frac{(x-1)^2}{x^3}\,dx &= \int \frac{x^2 - 2x + 1}{x^3}\,dx \\ &= \int (x^{-1} - 2x^{-2} + x^{-3})\,dx \\ &= \log|x| + \frac{2}{x} - \frac{1}{2x^2} + C.\end{aligned}$$

c. Dividing and integrating termwise gives

$$\begin{aligned}\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \frac{\sin^2 x + \cos x}{\sin^2 x}\,dx &= \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} (1 + \csc x \cot x)\,dx \\ &= (x - \csc x)\Big|_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \\ &= \left(\tfrac{1}{2}\pi - 1\right) - \left(\tfrac{1}{4}\pi - \sqrt{2}\right) \\ &= \tfrac{1}{4}\pi + \sqrt{2} - 1.\end{aligned}$$

d. Since $|x^2 - 9| = 9 - x^2$ on $[0, 3]$ and $|x^2 - 9| = x^2 - 9$ on $[3, 5]$, the interval additivity of the definite integral implies that

$$\begin{aligned}\int_0^5 |x^2 - 9|\,dx &= \int_0^3 (9 - x^2)\,dx + \int_3^5 (x^2 - 9)\,dx \\ &= \left(9x - \tfrac{1}{3}x^3\right)\Big|_0^3 + \left(\tfrac{1}{3}x^3 - 9x\right)\Big|_3^5 \\ &= \left\{(27 - 9) - 0\right\} + \left\{\left(\tfrac{125}{3} - 27\right) - (9 - 27)\right\} \\ &= \tfrac{98}{3}.\end{aligned}$$

15. If

$$y = \int_{\sqrt{x}}^1 \frac{t}{t^2+1}\,dt = -\int_1^{\sqrt{x}} \frac{t}{t^2+1}\,dt,$$

then by the Fundamental Theorem of Calculus and the Chain Rule,

$$\frac{dy}{dx} = -\frac{\sqrt{x}}{x+1} \cdot \frac{1}{2\sqrt{x}} = \frac{-1}{2(x+1)}.$$

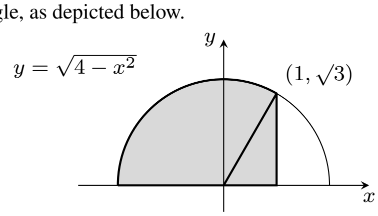
16. If $[\,1,3\,]$ is divided into n subintervals of equal length then the width of each subinterval is $2/n$ and the endpoints of the subintervals are $x_i = 1 + 2i/n$, for $0 \leqslant i \leqslant n$. The corresponding right endpoint sum is

$$\begin{aligned}\mathscr{R}_n &= \frac{2}{n}\sum_{i=1}^{n}\left\{\left(1+\frac{2i}{n}\right)^2 - \left(1+\frac{2i}{n}\right) + 3\right\}\\ &= 2\sum_{i=1}^{n}\left\{\frac{4i^2}{n^3} + \frac{2i}{n^2} + \frac{3}{n}\right\}\\ &= 2\left\{\frac{4}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{2}{n^2}\cdot\frac{1}{2}n(n+1) + 3\right\}\\ &= 2\left\{\frac{2}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \left(1+\frac{1}{n}\right) + 3\right\},\end{aligned}$$

and so

$$\int_1^3 (x^2 - x + 3)\,dx = \lim_{n\to\infty}\mathscr{R}_n = 2\left\{\tfrac{4}{3} + 1 + 3\right\} = \tfrac{32}{3}.$$

17. The integral in question is equal to the sum of the area of a circular sector and the area of a triangle, as depicted below.



The radius of the sector is 2 and its central angle is $\frac{2}{3}\pi$, so its area is equal to $\frac{1}{2}(2)^2\,\frac{2}{3}\pi = \frac{4}{3}\pi$. The base of the triangle is 1 and its height is $\sqrt{3}$, so its area is equal to $\frac{1}{2}(1)(\sqrt{3}) = \frac{1}{2}\sqrt{3}$. Therefore,

$$\int_{-2}^{1}\sqrt{4-x^2}\,dx = \tfrac{4}{3}\pi + \tfrac{1}{2}\sqrt{3}.$$

18. a. If $f(x) = \sqrt{x}$ then the domain of f is $[\,0,\infty\,)$ and

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\frac{1}{2\sqrt{x}} = 0,$$

but $f(x)\to\infty$ as $x\to\infty$, so the graph of f does not have a horizontal asymptote. So the graph of f need not have a horizontal asymptote under the given condition. (Many other examples are possible; *e.g.*, $\ln x$, or x^{α}, where $0<\alpha<1$.)

b. If f is a rational function then by division of polynomials, $f = p + r$, where p is a polynomial function and r is a proper rational function (p, or r, or both, may be identically zero). Then $r(x)\to 0$ as $x\to\infty$ (since r is proper) and so $r'(x)\to 0$ as $x\to\infty$ (since the sign of r' is eventually constant). Since $f' = p' + r'$, this implies that p' (since it is a polynomial) is identically zero, so p is a constant and therefore its graph is a horizontal asymptote of the graph of f. So the given condition does imply that the graph of a rational function must have a horizontal asymptote. (Other arguments are possible; *e.g.*, by considering the degrees of the numerator and denominator of $f(x)$ and $f'(x)$, or by considering the behaviour of $f(1/x)$ near zero.)