1. Given the graph of f below, evaluate each of the following expressions. Use ∞, $-\infty$, or "does not exist" as appropriate.

a. $\lim\limits_{x\to 2^-} f(x)$

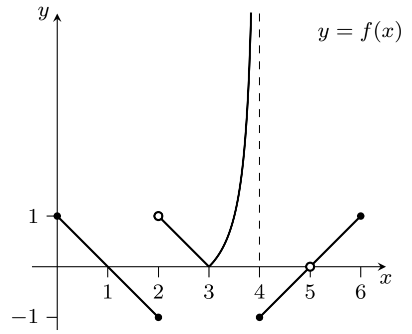
b. $\lim\limits_{x\to 2^+} f(x)$

c. $f(2)$

d. $\lim\limits_{x\to 4^-} f(x)$

e. $\lim\limits_{x\to 5^-} \dfrac{1}{f(x)}$

f. $\lim\limits_{x\to\infty} f(1/x)$



2. Evaluate each of the following limits.

a. $\lim\limits_{x\to 2} \dfrac{3x^2-5x-2}{x^2-4}$ b. $\lim\limits_{x\to\infty} \left\{x-\sqrt{x^2+4x}\,\right\}$

c. $\lim\limits_{\vartheta\to 0} \dfrac{\sin^2(5\vartheta)}{\vartheta^3-\vartheta^2}$ d. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{9x^2+6x+1}}{6x+1}$ e. $\lim\limits_{x\to 2^-} x\csc(\pi x)$

3. Find the derivative of $f(x) = 2x^2 + x$ using the definition of the derivative.

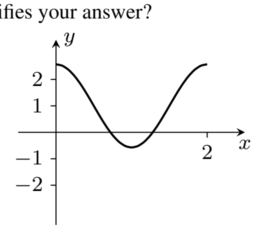
4. Let

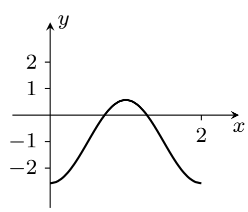
$$f(x) = \begin{cases} \dfrac{|x-3|}{x^2-9} & \text{if } x < 3,\, x \neq -3, \quad \text{and} \\ c & \text{if } x \geqslant 3. \end{cases}$$

Find all values of c for which the function f is continuous at 3.

5. Show that $f(x) = 3^x - x - 3$ has at least one real root in $(\,0,\infty\,)$.

6. Suppose that f is a function which is continuous on the interval $[\,0,2\,]$ and differentiable on the interval $(\,0,2\,)$. Suppose further that $f(0) = -1$ and $f(2) = 1$. One of the following two graphs is the graph of f'. Which one is it? What theorem justifies your answer?





7. For each of the following, find $\dfrac{dy}{dx}$.

a. $y = \dfrac{x^4}{3} + \dfrac{10}{\sqrt[5]{x^2}} + 2^x - \ln 7$ b. $y = x^7 \ln x$ c. $y = \dfrac{\sin(2x)\sqrt{x^4+5}}{(3x+1)^3}$

d. $y = (1+x^2)^{\cos x}$ e. $\ln(x+y) = 1 + \dfrac{1}{x^2}$

8. Find the coordinates of all points on the curve defined by $x^2 + xy + y^2 = 4$ at which the tangent line is parallel to the line defined by $y = x + 4$.

9. Find the equation of the tangent line to the graph of

$$y = \frac{3x}{x^2+2}$$

at the point whose x-coordinate is 1.

10. Find the absolute extrema of $f(x) = 3(x^2-2x)^{2/3}$ on the interval $[\,1,4\,]$.

11. A hot air balloon rising straight up from a level field is tracked by a range finder 1500 metres from the liftoff point. At the moment the range finder's angle of elevation is $\frac{1}{4}\pi$, that angle is increasing at a rate of $\frac{1}{5}$ radians per minute. How fast is the balloon rising at that moment?

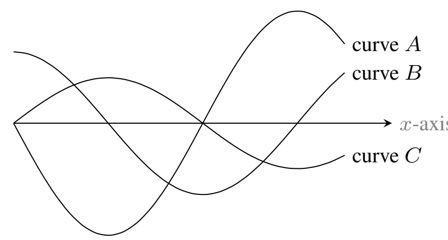
12. The cross section of a tunnel has the form of a rectangle surmounted by a semicircle. The perimeter of this cross section is 18 metres. Find the dimensions of the cross section which will maximize its area.

13. Sketch the graph of the function f defined by

$$f(x) = \frac{x+1}{(x-3)^2}, \quad \text{given} \quad f'(x) = -\frac{x+5}{(x-3)^3} \quad \text{and} \quad f''(x) = \frac{2(x+9)}{(x-3)^4}.$$

Be sure that your solution includes the domain of f, and all intercepts, asymptotes, intervals of monotonicity and concavity, local and global extrema and points of inflection.

14. In the figure below are the graphs of a function f, and its derivatives f' and f'', on some interval. Determine, with justification, which curve is the graph of f, which curve is the graph of f' and which curve is the graph of f''.



15. Given that $f'(x) = x + 2e^x$ and $f(0) = 5$, find $f(x)$.

16. Estimate the definite integral

$$\int_0^2 x\, 2^{2x}\, dx$$

from above and below using Riemann sums with a partition of $[\,0,2\,]$ into four subintervals of equal length.
For extra credit, compute the exact value of the integral as a limit of Riemann sums.

17. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{x^3-3x+2}{x^2}\, dx$ b. $\displaystyle\int \left(e^t + \frac{1}{\sqrt{4t}}\right) dt$

c. $\displaystyle\int_0^{\frac{1}{6}\pi} (\sec x)(\tan x + \cos^2 x)\, dx$ d. $\displaystyle\int_{-1}^{3} \big(|x| - 1\big)\, dx$

18. Find the derivative with respect to x of

$$F(x) = \int_0^{x^2} \frac{t}{1+e^t}\, dt.$$

19. In each part, give an example of a function which fits the description.

a. f is continuous everywhere and f' has a jump discontinuity.

b. f is continuous everywhere and f' has a infinite discontinuity.

Can you give an example in Part a such that f is differentiable everywhere? Why or why not?

1. Inspecting the graph of f reveals that a. $\lim_{x\to 2^-} f(x) = -1$, b. $\lim_{x\to 2^+} f(x) = 1$, c. $f(2) = -1$, d. $\lim_{x\to 4^-} f(x) = \infty$, e. $\lim_{x\to 5^-} \{1/f(x)\} = -\infty$ (since $f(x) \to 0^-$ as $x \to 5^-$) and f. $\lim_{x\to\infty} f(1/x) = 1$ (since $t = 1/x \to 0^+$ as $x \to \infty$ and $f(t) \to 1$ as $t \to 0^+$).

2. a. Since $3x^2 - 5x - 2 = (3x+1)(x-2)$ and $x^2 - 4 = (x+2)(x-2)$, it follows that

$$\lim_{x\to 2} \frac{3x^2 - 5x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{3x+1}{x+2} = \tfrac{7}{4}.$$

b. Since (rationalizing the numerator)

$$x - \sqrt{x^2+4x} = \frac{-4x}{x+\sqrt{x^2+4x}} = \frac{-4}{1+\sqrt{1+4/x}}$$

if $x > 0$, it follows that

$$\lim_{x\to\infty} \{x - \sqrt{x^2+4x}\} = \lim_{x\to\infty} \frac{-4}{1+\sqrt{1+4/x}} = \frac{-4}{1+1} = -2.$$

c. Factorizing the denominator and then multiplying and dividing by 25 gives

$$\lim_{\vartheta\to 0} \frac{\sin^2(5\vartheta)}{\vartheta^3 - \vartheta^2} = \lim_{\vartheta\to 0} \left\{ \left(\frac{\sin(5\vartheta)}{5\vartheta}\right)^2 \cdot \frac{25}{\vartheta - 1} \right\} = 1 \cdot 1 \cdot \frac{25}{0-1} = -25.$$

d. If $x < 0$ then $\sqrt{x^2} = -x$ and

$$\frac{\sqrt{9x^2+6x+1}}{6x+1} = -\frac{\sqrt{9+6/x+1/x^2}}{6+1/x},$$

which implies that

$$\lim_{x\to -\infty} \frac{\sqrt{9x^2+6x+1}}{6x+1} = -\tfrac{\sqrt{9}}{6} = -\tfrac{1}{2}.$$

e. As $x \to 2^-$, $\sin(\pi x) \to 0^-$, and so

$$\lim_{x\to 2^-} x\csc(\pi x) = \lim_{x\to 2^-} \frac{x}{\sin(\pi x)} = -\infty.$$

3. If $f(x) = 2x^2 + x$ then

$$f(x+h) = 2(x+h)^2 + (x+h) = 2x^2 + 4xh + 2h^2 + x + h;$$

so $f(x+h) - f(x) = 4xh + 2h^2 + h = h(4x + 2h + 1)$, and therefore

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} (4x + 2h + 1) = 4x + 1.$$

4. First notice that $x^2 - 9 = (x-3)(x+3)$ and $|x-3| = -(x-3)$ if $x < 3$, and so

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x-3|}{x^2-9} = \lim_{x\to 3^-} \frac{-1}{x+3} = \tfrac{1}{6}.$$

On the other hand

$$f(3) = \lim_{x\to 3^+} f(x) = c.$$

Therefore, f is continuous at 3 if, and only if, $c = -\frac{1}{6}$.

5. The function f is differentiable (and therefore also continuous) on $\mathbb{R}$, so the Intermediate Value Theorem and the Mean Value Theorem apply to f on any closed interval of positive length. Since $f(1) = 3 - 1 - 3 = -1 < 0$ and $f(2) = 3^2 - 2 - 3 = 4 > 0$, the Intermediate Value Theorem implies that there is a real number r such that $1 < r < 2$ and $f(r) = 0$. This shows that f has a positive real zero, as required.

In fact, f has a exactly one positive real zero. For, if $0 < r_1 < r_2$, $f(r_1) = 0$ and $f(r_2) = 0$, then the Mean Value Theorem implies that there is a real number s such that $r_1 < s < r_2$ and $f'(s) = 0$. But $s > 0$ (since $r_1 > 0$), which implies that $f'(s) = 3^s \log 3 - 1 > \log 3 - 1 > 0$. So there cannot be r_1, r_2 as described above; *i.e.*, f cannot have two different positive real zeros. This shows that f has a unique positive real zero.

6. The Mean Value Theorem implies (since f satisfies its hypotheses) that there is a real number ξ such that $0 < \xi < 2$ and $f(2) - f(0) = f'(\xi)(2-0)$, *i.e.*, $2 = f'(\xi)\cdot 2$, or $f'(\xi) = 1$. Since the graph on the right lies entirely below the line $y = 1$, the graph of f' must be the graph on the left.

7. a. Writing $y = \frac{1}{3}x^4 + 10x^{-2/5} + 2^x - \log 7$ gives

$$\frac{dy}{dx} = \tfrac{4}{3}x^3 - 4x^{-7/5} + 2^x \log 2.$$

b. If $y = x^7 \log x$ then the Product Rule implies that

$$\frac{dy}{dx} = 7x^6 \log x + x^6 = x^6(7\log x + 1).$$

c. Since $\frac{dy}{dx} = y\frac{d}{dx}\{\log|y|\}$, one has

$$\frac{dy}{dx} = y\frac{d}{dx}\left\{\log|\sin(2x)| + \tfrac{1}{2}\log(x^4+5) - 3\log|3x+1|\right\}$$
$$= \frac{\sin(2x)\sqrt{x^4+5}}{(3x+1)^3}\left\{2\cot(2x) + \frac{2x^3}{x^4+5} - \frac{9}{3x+1}\right\}.$$

d. As in the previous part of this question

$$\frac{dy}{dx} = y\frac{d}{dx}\{\cos x \log(1+x^2)\}$$
$$= (1+x^2)^{\cos x}\left\{\frac{2x\cos x}{1+x^2} - \sin x\,\log(1+x^2)\right\}.$$

e. Implicit differentiation gives

$$\frac{1}{x+y}\left\{1 + \frac{dy}{dx}\right\} = -\frac{2}{x^3}, \qquad \text{and so} \qquad \frac{dy}{dx} = -1 - \frac{2(x+y)}{x^3}.$$

8. It is required to find all pairs (x, y) where $x^2 + xy + y^2 = 4$ and $\frac{dy}{dx} = 1$. Implicit differentiation gives

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0, \qquad \text{or} \qquad x + y = 0, \qquad \text{if} \qquad \frac{dy}{dx} = 1.$$

So $0 = (x+y)^2 = x^2 + 2xy + y^2$ which, together with the equation of the curve, implies that $xy = -4$, and so (since $x + y = 0$), $x^2 = 4$ or $x = \pm 2$ and $y = \mp 2$. Checking the equation defining $\frac{dy}{dx}$ above shows that the required points are $(2, -2)$ and $(-2, 2)$.

9. If

$$y = \frac{3x}{x^2+2}, \qquad \text{then} \qquad \frac{dy}{dx} = \frac{3\big(x^2 + 2 - x(2x)\big)}{(x^2+2)^2} = \frac{3(2-x^2)}{(x^2+2)^2},$$

by the Quotient Rule. If $x = 1$, then $y = 1$ and $\frac{dy}{dx} = \frac{1}{3}$, so $y = 1 + \frac{1}{3}(x-1)$ is an equation of the line tangent to the given curve at the point $(1, 1)$.

10. If f is continuous on $\mathbb{R}$, so by the Extreme Value Theorem it has a largest and smallest value on $[1, 4]$. If

$$f(x) = 3(x^2 - 2x)^{2/3} = 3\big(x(x-2)\big)^{2/3},$$

then

$$f'(x) = 2(x^2 - 2x)^{-1/3}(2x - 2) = 4\big(x-1\big)\big(x(x-2)\big)^{-1/3},$$

so the critical numbers of f are 0, 1 and 2, of which only 2 belongs so $(1, 4)$. Now $f(1) = 3$, $f(2) = 0$ and $f(4) = 12$, so the absolute maximum and minimum values of f on $[1, 4]$ are, respectively, 12 and 0.

11. Let y be the altitude (in kilometres) of the balloon, and let ϑ be the angle (in radians) of elevation of the range finder. Then $y = \frac{3}{2}\tan\vartheta$, and so

$$\frac{dy}{dt} = \tfrac{3}{2}\sec^2\vartheta\,\frac{d\vartheta}{dt}.$$

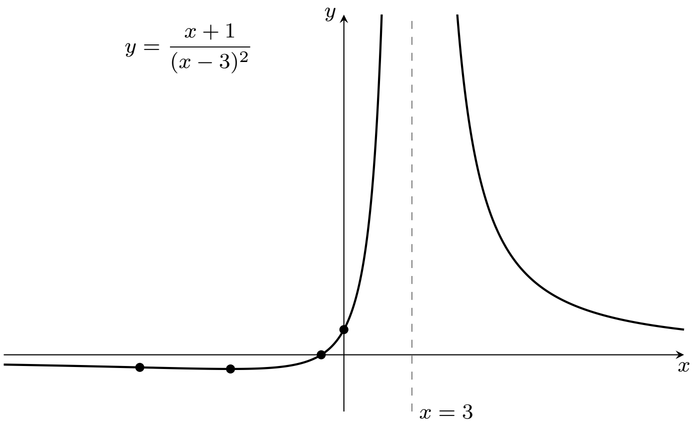
When $\vartheta = \frac{1}{4}\pi$, $\frac{d\vartheta}{dt} = \frac{1}{5}$ and $\sec^2\vartheta = 2$, so $\frac{dy}{dt} = \frac{3}{5}$. Hence, the balloon is rising at a rate of $\frac{3}{5}$ kilometres per minute when the range finder's angle of elevation is half a right angle.

12. For the cross section of the tunnel, let r be the radius of the semicircle and let h be the height of the rectangle (each measured in metres). Then $18 = 2h + (\pi+2)r$ and it is required to find h and r such that the area

$$A = 2rh + \tfrac{1}{2}\pi r^2 = r\big(18 - (\pi+2)\big) + \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}r\big(36 - (\pi+4)r\big).$$

Since the graph of A as a function of r is a downward opening parabola, the value of A is maximized if r is the average of the parabola's intercepts, *i.e.*, $r = 18/(\pi+4)$. Then $h = 9 - 9(\pi+2)/(\pi+4) = r$. Hence, the area of the cross section of the tunnel is largest if the radius of the semicircle and the height of the rectangle are each equal to $18/(\pi+4)$ metres.

13. The domain of f is $\mathbb{R} \setminus \{3\}$, and the intercepts of the graph of f are $\left(0, \frac{1}{9}\right)$ and $(-1, 0)$. Since $\lim_{x\to\pm\infty} f(x) = 0$, the x-axis is the horizontal asymptote of the graph of f. Since $\lim_{x\to 3^\pm} f(x) = \infty$ (and otherwise f is continuous) the line defined by $x = 3$ is the vertical asymptote of the graph of f, and f has no global maximum. Next, $f'(x)$ is negative if $x < -5$ or if $x > 3$, and $f'(x)$ is positive if $-5 < x < 3$, so f is increasing on $(-5, 3)$, and is decreasing on $(-\infty, -5)$ and on $(3, \infty)$, and therefore $f(-5) = -\frac{1}{16}$ is the local (and global) minimum value of f. Finally, $f''(x)$ is negative if $x < -9$ and is positive if $x > -9$ and $x \neq 3$, so the graph of f is concave down on $(-\infty, -9)$, and is concave up on $(-9, 3)$ and on $(3, \infty)$, and $\left(-9, -\frac{1}{18}\right)$ is the inflection point of the graph of f. Below is a sketch (not to scale; the y-axis is dilated by a factor of 10) of the graph of f, with the points of interest emphasized.



14. Let a, b, c denote, respectively, the function whose graph is Curve A, B, C. Where b is positive, c is increasing, where b is negative, c is decreasing, and where b is zero, Curve C has a horizontal tangent line. Likewise, a bears the same relation to b as b does to c. So Curve C is the graph of f, Curve B is the graph of f' and Curve A is the graph of f''.

15. If $f'(x) = x + 2e^x$ then $f(x) = \frac{1}{2}x^2 + 2e^x + C$ for some real number C. If $f(0) = 5$ then $0 + 2e^0 + C = 5$, which implies that $C = 3$. Therefore, $f(x) = \frac{1}{2}x^2 + 2e^x + 3$.

16. If $[0, 2]$ is divided into four subintervals of equal length then the endpoints of the subintervals are $x_i = \frac{1}{2}i$, for $0 \leqslant i \leqslant 4$. If $f(x) = x2^{2x}$, then f is increasing on $[0, 2]$, so a lower estimate is obtained by evaluating at left endpoints and an upper estimate is obtained by evaluating at right endpoints. So

$$\mathscr{L}_4 = \tfrac{1}{2}\left\{f(0) + f\left(\tfrac{1}{2}\right) + f(1) + f\left(\tfrac{3}{2}\right)\right\} = \tfrac{1}{2}\left\{0 + 1 + 4 + 12\right\} = \tfrac{17}{2}$$

is a lower estimate of the integral and

$$\mathscr{R}_4 = \tfrac{1}{2}\left\{f\left(\tfrac{1}{2}\right) + f(1) + f\left(\tfrac{3}{2}\right) + f(2)\right\} = \tfrac{1}{2}\left\{1 + 4 + 12 + 32\right\} = \tfrac{49}{2}$$

is an upper estimate of the integral.

If $[0, 2]$ is divided into n subintervals of equal length, then the width of each subinterval is $2/n$ and the endpoints of the subintervals are $2i/n$, for $0 \leqslant i \leqslant n$. The Riemann sum obtained by evaluating at right endpoints is

$$\mathscr{R}_n = \frac{2}{n}\sum_{i=1}^{n}\left\{\frac{2i}{n}\,2^{2(2i/n)}\right\} = \frac{4}{n^2}\sum_{i=1}^{n} i\left(2^{4/n}\right)^i.$$

The sum of a geometric progression gives

$$\alpha^i + \alpha^{i+1} + \cdots + \alpha^n = \frac{\alpha^i(\alpha^{n-i+1} - 1)}{\alpha - 1} = \frac{\alpha^{n+1} - \alpha^i}{\alpha - 1},$$

and therefore also

$$\sum_{i=1}^{n} i\alpha^i = \sum_{i=1}^{n}(\alpha^i + \alpha^{i+1} + \cdots + \alpha^n) = \sum_{i=1}^{n}\frac{\alpha^{n+1} - \alpha^i}{\alpha - 1}$$
$$= \frac{n\alpha^{n+1}}{\alpha - 1} - \frac{\alpha}{\alpha - 1}\sum_{i=0}^{n-1}\alpha^i = \frac{n\alpha^{n+1}}{\alpha - 1} - \frac{\alpha(\alpha^n - 1)}{(\alpha - 1)^2}.$$

So (with $\alpha = 2^{4/n}$)

$$\mathscr{R}_n = \frac{4}{n}\cdot\frac{2^{4(1+1/n)}}{2^{4/n} - 1} - \frac{4}{n^2}\cdot\frac{2^{4/n}(2^4 - 1)}{(2^{4/n} - 1)^2}.$$

Since

$$\lim_{n\to\infty}\left\{n(2^{4/n} - 1)\right\} = 4\log 2\lim_{t\to 0^+}\frac{e^t - 1}{t} = 4\log 2,$$

where $t = (4\log 2)/n$, by the basic indeterminate limit for exponential functions, and since $a/n \to 0$ as $n \to \infty$ for any real number a, it follows that

$$\int_0^2 x2^{2x}\,dx = \lim_{n\to\infty}\mathscr{R}_n = \frac{4\cdot 2^4}{4\log 2} - \frac{4(2^4 - 1)}{(4\log 2)^2} = \frac{16}{\log 2} - \frac{15}{4(\log 2)^2}.$$

17. a. Termwise division and integration yields

$$\int\frac{x^3 - 3x + 2}{x^2}\,dx = \int\left(x - 3x^{-1} + 2x^{-2}\right)dx$$
$$= \tfrac{1}{2}x^2 - 3\log|x| - 2/x + C.$$

b. Revising the second term and integrating termwise yields

$$\int\left(e^t + \frac{1}{\sqrt{4t}}\right)dt = \int\left(e^t + \tfrac{1}{2}t^{-1/2}\right)dt = e^t + \sqrt{t} + C.$$

c. Expanding the integrand and then integrating termwise yields

$$\int_0^{\frac{1}{6}\pi}(\sec x)(\tan x + \cos^2 x)\,dx = \int_0^{\frac{1}{6}\pi}(\sec x\tan x + \cos x)\,dx$$
$$= (\sec x + \sin x)\Big|_0^{\frac{1}{6}\pi}$$
$$= \left(\tfrac{2}{3}\sqrt{3} + \tfrac{1}{2}\right) - (1 + 0)$$
$$= \tfrac{2}{3}\sqrt{3} - \tfrac{1}{2}.$$

d. Since $|x| = -x$ if $x \leqslant 0$, it follows that

$$\int_{-1}^{0}\left(|x| - 1\right)dx = -\int_{-1}^{0}(x + 1)\,dx = -\left(\tfrac{1}{2}x^2 + x\right)\Big|_{-1}^{0} = -\tfrac{1}{2}.$$

Likewise, since $|x| = x$ if $x \geqslant 0$, it follows that

$$\int_0^3\left(|x| - 1\right)dx = \int_{-1}^{0}(x - 1)\,dx = \left(\tfrac{1}{2}x^2 - x\right)\Big|_0^3 = \tfrac{3}{2}.$$

Therefore, by the interval additivity of the definite integral,

$$\int_{-1}^{3}\left(|x| - 1\right)dx = -\tfrac{1}{2} + \tfrac{3}{2} = 1.$$

(This integral could also have been evaluated by interpreting it in terms of areas of triangles.)

18. By the (first form of the) Fundamental Theorem of Calculus and the Chain Rule,

$$F'(x) = \frac{x^2}{1 + e^{x^2}}\cdot 2x = \frac{2x^3}{1 + e^{x^2}}.$$

19. a. If $f(x) = \sqrt{x^2}$ then f' has a jump discontinuity at the origin since $f'(x) = -1$ if $x < 0$ and $f'(x) = 1$ if $x > 0$ (and $f'(0)$ is undefined).
b. If $f(x) = \sqrt[3]{x^2}$ then f' has an infinite discontinuity at the origin since $f'(x) \to -\infty$ as $x \to 0^-$ and $f'(x) \to \infty$ as $x \to 0^+$.

There is no example for Part a if f is required to be differentiable everywhere. For if a is any real number and $x < a$ then the Mean Value Theorem implies that there is a real number ξ such that $x < \xi < a$ and

$$\frac{f(x) - f(a)}{x - a} = f'(\xi),$$

which implies that if $\lim_{x\to a^-} f'(x)$ and $f'(a)$ are defined then they must be equal. The same argument applies to the limit of $f'(x)$ as $x \to a^+$.