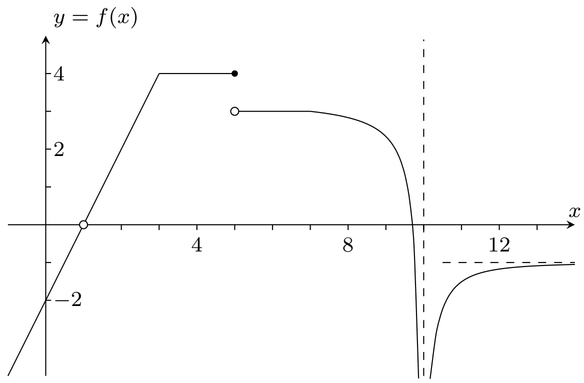
1. Below is the graph of a function f.



Evaluate each of the following. Use ∞, $-\infty$ or "does not exist" where appropriate.

a. $\lim_{x\to 1} f(x)$ b. $\lim_{x\to 10} f(x)$ c. $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$

d. $\lim_{x\to\infty} f\left(\frac{1}{x}\right)$ e. $\lim_{x\to\infty} f(x)$ f. $f'(5)$

2. Evaluate each of the following limits.

a. $\lim_{x\to 2} \dfrac{x^3-7x^2+10x}{2x^2-3x-2}$ b. $\lim_{x\to 5} \dfrac{\frac{1}{x-8}+\frac{1}{3}}{x^2-25}$ c. $\lim_{x\to 0} \dfrac{\sin(5x)}{x^2+10x}$

d. $\lim_{x\to \frac{3}{2}\pi} \dfrac{\cos x}{1-\sqrt{1-\cos x}}$ e. $\lim_{x\to\infty} \sqrt[3]{\dfrac{7+x^2-8x^3}{x^3-x+\pi}}$ f. $\lim_{x\to 2^-} \dfrac{|x-2|}{(x-2)^2}$

3. Find all values of a and b so that the function f, defined by

$$f(x) = \begin{cases} ax-b & \text{if } x \leqslant -1, \\ 2x^2+3ax+b & \text{if } -1 < x \leq 1 \text{ and} \\ 4 & \text{if } x > 1, \end{cases}$$

is continuous on $\mathbb{R}$.

4. Use the limit definition of the derivative to find $f'(x)$, where $f(x) = x + \dfrac{1}{x}$.

5. Find $\dfrac{dy}{dx}$ for each of the following.

a. $y = \dfrac{x^2 - \sqrt[3]{x^4} + \pi\sqrt{x}}{\sqrt{x}}$ b. $y = e^{5x^2} - 6x\,4^x - 3\ln(7x+1) - \log_2(\cos x)$

c. $y = \left(\dfrac{x^2+2}{x^2-2}\right)^{10}$ d. $y = \dfrac{\sqrt{x^2+2}\,\sqrt[3]{x^3+3}}{\sqrt[4]{x^4+4}}$ e. $e^{xy} + 7 = y\tan x$

6. How many tangent lines to the graph of

$$f(x) = \frac{x}{2x-1}$$

contain the point $(-7, 1)$? At which points do these tangent lines touch the curve?

7. Prove that the equation $e^x = -x + 2$ has *exactly* one real root.

8. Two sides of a triangle are 2 cm and 5 cm long respectively, and the angle between them is increasing at a rate of $\frac{1}{2}$ rad/s. At what rate is the area of the triangle is increasing when the angle between the sides is $\frac{\pi}{3}$?

9. Find the absolute extrema of $f(x) = \dfrac{2x}{e^x}$ on $[0, 10]$.

10. If the function f is defined by

$$f(x) = (x+3)^{1/3}(x-1)^{2/3},$$

then

$$f'(x) = \frac{3x+5}{3(x+3)^{2/3}(x-1)^{1/3}} \quad \text{and} \quad f''(x) = \frac{-32}{9(x+3)^{5/3}(x-1)^{4/3}}.$$

Sketch the graph of f. Make sure that your solution includes the domain of f, and all asymptotes, intercepts, intervals of monotonicity and concavity, extrema and points of inflection.

11. Prove that if $f > 0$ and $f' < 0$ on $\mathbb{R}$, then the function g defined by

$$g(x) = \frac{1-f(x)}{1+f(x)}$$

is increasing on $\mathbb{R}$.

12. An open-topped box is made by cutting squares from the corners of a square piece of sheet metal and folding up the sides. If the volume of the box is $2\,\text{m}^3$ and the area of the original square sheet is as small as possible, what are the dimensions of the box?

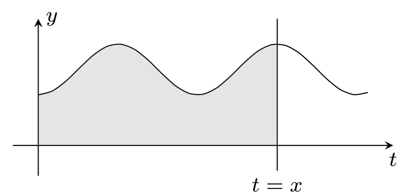
13. Find $f(x)$ if $f''(x) = -\sin x$, $f(0) = -1$ and $f(\frac{\pi}{2}) = \pi$.

14. Evaluate each of the following integrals.

a. $\displaystyle\int \left(x^5 + 5^x + \ln 5\right)\,dx$ b. $\displaystyle\int_1^e \left(1-\frac{1}{x}\right)^2 dx$

c. $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \tan x\,(\cos x - \sec x)\,dx$ d. $\displaystyle\int \left(\sqrt{2x} + 2x\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

15. Let $g(x)$ be the area of the region enclosed by the graph of $y = 1 + \sin^2(t)$, the t-axis, the y-axis, and the line $t = x$, as shown below. Find $g'(x)$.



16. Evaluate the integral

$$\int_0^1 (4x - x^2)\,dx$$

as a limit of Riemann sums. (No credit is earned if you use the Fundamental Theorem of Calculus to evaluate the integral.)

1. Inspecting the graph of f reveals that

a. $\lim_{x\to 1} f(x) = 0$,
b. $\lim_{x\to 10} f(x) = -\infty$,

c. $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = 2$,
d. $\lim_{x\to\infty} f(1/x) = \lim_{t\to 0^+} f(t) = -2$,

e. $\lim_{x\to\infty} f(x) = -1$ and
f. $f'(5)$ is undefined.

2. a. Since
$$\frac{x^3-7x^2+10x}{2x^2-3x-2} = \frac{x(x-2)(x-5)}{(2x+1)(x-2)} = \frac{x(x-5)}{2x+1},$$
provided $x \neq 2$,
$$\lim_{x\to 2}\frac{x^3-7x^2+10x}{2x^2-3x-2} = \lim_{x\to 2}\frac{x(x-5)}{2x+1} = \frac{2(2-5)}{2(2)+1} = -\tfrac{6}{5}.$$
b. Since
$$\frac{\frac{1}{x-8}+\frac{1}{3}}{x^2-25} = \frac{x-5}{3(x-8)}\cdot\frac{1}{(x+5)(x-5)} = \frac{1}{3(x+5)(x-8)},$$
provided $x \neq 5$,
$$\lim_{x\to 5}\frac{\frac{1}{x-8}+\frac{1}{3}}{x^2-25} = \lim_{x\to 5}\frac{1}{3(x+5)(x-8)} = \frac{1}{3(5+5)(5-8)} = -\tfrac{1}{90}.$$
c. Factorizing the denominator of the expression in the limit, and then multiplying and dividing by 5, yields
$$\lim_{x\to 0}\frac{\sin(5x)}{x^2+10x} = \lim_{x\to 0}\left\{\frac{\sin(5x)}{5x}\cdot\frac{5}{x+10}\right\} = 1\cdot\frac{5}{0+10} = \tfrac{1}{2}.$$
since $(\sin t)/t \to 1$ as $t \to 0$.

d. Multiplying and dividing by $1+\sqrt{1-\cos x}$ gives
$$\frac{\cos x}{1-\sqrt{1-\cos x}} = \frac{(\cos x)(1+\sqrt{1-\cos x}\,)}{1-(1-\cos x)} = 1+\sqrt{1-\cos x},$$
provided $x \neq \frac{1}{2}\pi + k\pi$ for any integer k, and therefore
$$\lim_{x\to\frac{3}{2}\pi}\frac{\cos x}{1-\sqrt{1-\cos x}} = \lim_{x\to\frac{3}{2}\pi}\left(1+\sqrt{1-\cos x}\,\right) = 1+\sqrt{1-0} = 2.$$
e. Extracting the dominant powers of x from the expression in the limit gives
$$\lim_{x\to\infty}\sqrt[3]{\frac{7+x^2-8x^3}{x^3-x+\pi}} = \lim_{x\to\infty}\sqrt[3]{\frac{7/x^3+1/x-8}{1-1/x^2+\pi/x^3}} = \sqrt[3]{(-8)} = -2,$$
since $1/x^n \to 0$ as $x\to\infty$ for any positive integer n.

f. Since $(x-2)^2 = |x-2|^2$ and $|x-2| \to 0^+$ as $x \to 2-$, it follows that
$$\lim_{x\to 2^-}\frac{|x-2|}{(x-2)^2} = \lim_{x\to 2^-}\frac{1}{|x-2|} = \infty.$$

3. Since f is equal to a polynomial function on $(-\infty,-1)$, f is continuous there, and for the same reason f is continuous on $(-1,1)$ and on $(1,\infty)$, regardless of the values of a and b. Now
$$\lim_{x\to -1^-} f(x) = f(-1) = -a-b \qquad\text{and}\qquad \lim_{x\to -1^+} f(x) = 2-3a+b,$$
so f is continuous at -1 if, and only if, $-a-b = 2-3a+b$, or $a = b+1$. Also,
$$\lim_{x\to 1^-} f(x) = f(1) = 2+3a+b = 4b+5 \qquad\text{and}\qquad \lim_{x\to 1^+} f(x) = 4,$$
so f is continuous at 1 if, and only if, $4b+5 = 4$, or $b = -\frac{1}{4}$, and hence $a = \frac{3}{4}$. Therefore, f is continuous on $\mathbb{R}$ if, and only if, $a = \frac{3}{4}$ and $b = -\frac{1}{4}$.

4. If $f(x) = x + 1/x$, then
$$f(x+h)-f(x) = x+h+\frac{1}{x+h}-x-\frac{1}{x} = h-\frac{h}{x(x+h)},$$
and so
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left\{1-\frac{1}{x(x+h)}\right\} = 1-\frac{1}{x^2}.$$

5. a. If
$$y = \frac{x^2-\sqrt[3]{x^4}+\pi\sqrt{x}}{\sqrt{x}} = x^{3/2}-x^{5/6}+\pi,$$
then
$$\frac{dy}{dx} = \tfrac{3}{2}x^{1/2}-\tfrac{5}{6}x^{-1/6}.$$
b. If
$$\begin{aligned} y &= e^{5x^2}-6x4^x-3\log(7x+1)-\log_2(\cos x)\\ &= e^{5x^2}-6xe^{x\log 4}-3\log(7x+1)-\log(\cos x)/\log 2,\end{aligned}$$
then
$$\begin{aligned}\frac{dy}{dx} &= e^{5x^2}10x-6\cdot 4^x-6x4^x\log 4-3\cdot\frac{7}{7x+1}+\frac{\sin x}{(\cos x)\log 2}\\ &= e^{5x^2}10x-6\cdot 4^x(1+x\log 4)-\frac{21}{7x+1}+(\tan x)/\log 2.\end{aligned}$$
c. If
$$y = \left(\frac{x^2+2}{x^2-2}\right)^{10},$$
then
$$\frac{dy}{dx} = 10\left(\frac{x^2+2}{x^2-2}\right)^9\cdot\frac{2x(x^2-2)-(x^2+2)2x}{(x^2-2)^2} = -\frac{80x(x^2+2)^9}{(x^2-2)^{11}}.$$
d. If
$$y = \frac{\sqrt{x^2+2}\,\sqrt[3]{x^3+3}}{\sqrt[4]{x^4+4}},$$
then
$$\begin{aligned}\frac{dy}{dx} &= y\frac{d}{dx}\log y = y\frac{d}{dx}\left\{\tfrac{1}{2}\log(x^2+2)+\tfrac{1}{3}\log(x^3+3)-\tfrac{1}{4}\log(x^4+4)\right\}\\ &= \frac{x\sqrt{x^2+2}\,\sqrt[3]{x^3+3}}{\sqrt[4]{x^4+4}}\left\{\frac{1}{x^2+2}+\frac{x}{x^3+3}-\frac{x^2}{x^4+4}\right\}.\end{aligned}$$
e. If $e^{xy}+7 = y\tan x$, then
$$e^{xy}\left(y+x\frac{dy}{dx}\right) = \frac{dy}{dx}\tan x + y\sec^2 x,$$
or
$$\left(xe^{xy}-\tan x\right)\frac{dy}{dx} = y(\sec^2 x - e^{xy}), \quad\text{and so}\quad \frac{dy}{dx} = \frac{y(\sec^2 x-e^{xy})}{xe^{xy}-\tan x}.$$

6. If
$$f(x) = \frac{x}{2x-1}, \qquad\text{then}\qquad f'(x) = -\frac{1}{(2x-1)^2},$$
and the equation of the tangent line at a point on the graph of f contains the point $(-7,1)$ where $f(x)-1 = f'(x)(x+7)$, *i.e.*,
$$\frac{x}{2x-1}-1 = -\frac{1}{(2x-1)^2}(x+7).$$
This last equation is equivalent to $x \neq \frac{1}{2}$ and $x(2x-1)-(2x-1)^2 = -(x+7)$, or $2x^2-x-4x^2+4x-1 = -x-7$, *i.e.*, $0 = 2x^2+4x-6 = 2(x+1)(x-3)$. So there are two points on the graph of f at which the tangent line passes through the point $(-7,1)$. The tangent line at the point $\left(-1,\frac{1}{3}\right)$ is defined by
$$y-\tfrac{1}{3} = -\tfrac{1}{9}(x+1), \qquad\text{or}\qquad x+9y = 2,$$
and the tangent line at the point $\left(3,\frac{3}{5}\right)$ is defined by
$$y-\tfrac{3}{5} = -\tfrac{1}{25}(x-3), \qquad\text{or}\qquad x+25y = 18.$$

7. The real solutions of the equation $e^x = -x+2$ are precisely the real zeros of the function f defined by $f(x) = e^x+x-2$. The function f is differentiable on $\mathbb{R}$, so the Intermediate Value Theorem and Rolle's Theorem apply to f on any closed interval of positive length. Since $f(0) = -1 < 0$ and $f(1) = e-1 > 0$, the Intermediate Value Theorem implies that there is a real number ξ such that $0 < \xi < 1$ and $f(\xi) = 0$. Hence, f has at least one real zero. If $\alpha < \beta$ are real numbers such that $f(\alpha) = 0$ and $f(\beta) = 0$, then by Rolle's Theorem there is a real number γ such that $\alpha < \gamma < \beta$ and $f'(\gamma) = 0$. But $f'(\gamma) = e^\gamma+1 > 1$, so there is no such pair α, β; *i.e.*, f does not have two real zeros. Therefore, f has exactly one real zero; *i.e.*, the equation $e^x = -x+2$ has exactly one real solution.

8. The area of the triangle in question is given by $A = \frac{1}{2}(5)(2\sin\vartheta) = 5\sin\vartheta$, where ϑ is the angle (measured in radians) between the sides of fixed length. Then

$$\frac{dA}{dt} = 5\cos\vartheta\,\frac{d\vartheta}{dt} = \tfrac{5}{2}\cos\vartheta, \qquad \text{and so} \qquad \left.\frac{dA}{dt}\right|_{\vartheta=\frac{1}{3}\pi} \tfrac{5}{2}\cdot\tfrac{1}{2} = \tfrac{5}{4}.$$

Therefore, the area of the triangle is increasing at a rate of $\frac{5}{4}$ square centimetres per second, when the angle between the fixed sides is $\frac{1}{3}\pi$.

9. If $f(x) = 2xe^{-x}$, then $f'(x) = 2\bigl(e^{-x} - xe^{-x}\bigr) = 2e^{-x}(1-x)$, so 1 is the only critical number of f in $(\,0,10\,)$. Comparing

$$f(0) = 0, \qquad f(1) = 2e^{-1} \qquad \text{and} \qquad f(10) = 20e^{-10},$$

reveals that the largest value of f on $[\,0,10\,]$ is $2e^{-1}$ and the smallest value of f on $[\,0,10\,]$ is 0. (That $2 < e < 3$, implies that $0 < 20e^{-10} < \frac{5}{256} < \frac{2}{3} < 2e^{-1}$.)

10. The domain of f is the set of all real numbers, on which f is continuous, so the graph of f has no vertical asymptotes. The intercepts of the graph of f are $(-3,0)$, $(0,\sqrt[3]{3})$ and $(1,0)$. Also, rationalizing the numerator of $f(x) - x$ and extracting dominant terms gives

$$\begin{aligned} f(x) - x &= \frac{(x+3)(x-1)^2 - x^3}{(x+3)^{2/3}(x-1)^{4/3} + x(x+3)^{1/3}(x-1)^{2/3} + x^2} \\ &= \frac{1 - 5/x + 3/x^2}{(1+3/x)^{2/3}(1-1/x)^{4/3} + (1+3/x)^{1/3}(1-1/x)^{2/3} + 1}, \end{aligned}$$

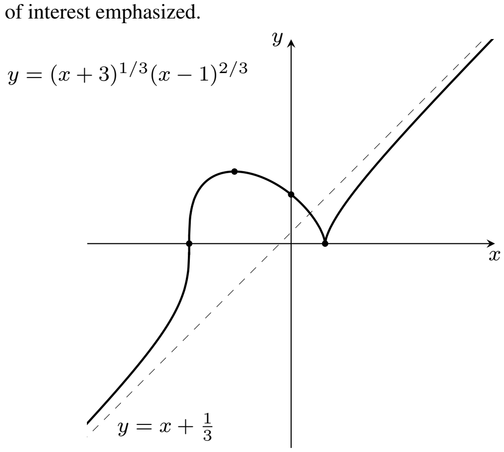
and so $f(x) - x \to \frac{1}{3}$ as $x \to \pm\infty$, which implies that $y = x + \frac{1}{3}$ is the oblique asymptote of the graph of f. It follows that f has no global extreme values.

The derivative $f'(x)$ is equal to zero if $x = -\frac{5}{3}$, and is undefined if $x = -3$ or $x = 1$. Otherwise, $f'(x)$ is negative if $-\frac{5}{3} < x < 1$ and is positive if $x < -3$ or $-3 < x < -\frac{5}{3}$ or $x > 1$. Therefore, f is decreasing on $\bigl(-\frac{5}{3},1\bigr)$, and is increasing on $\bigl(-\infty,-\frac{5}{3}\bigr)$, and on $\bigl(1,\infty\bigr)$; so

$$f\bigl(-\tfrac{5}{3}\bigr) = \bigl(-\tfrac{5}{3}+3\bigr)^{1/3}\bigl(-\tfrac{5}{3}-1\bigr)^{2/3} = \bigl(\tfrac{4}{3}\bigr)^{1/3}\bigl(-\tfrac{8}{3}\bigr)^{2/3} = \tfrac{4}{3}\sqrt[3]{4}$$

is a local maximum value of f, and $f(1) = 0$ is a local minimum value of f.

The second derivative $f''(x)$ is never equal to zero, is undefined if $x = -3$ or $x = 1$, is positive if $x < -3$ and is negative if $-3 < x < 1$ or $x > 1$. Therefore, the graph of f is concave up on $(\,-\infty,-3\,)$, is concave down on $(\,-3,1\,)$ and on $(\,1,\infty\,)$, and the only point of inflection of the graph of f is $(-3,0)$. The graph of f is sketched below, with the oblique asymptote drawn as a dashed line and the points of interest emphasized.



11. If

$$g(x) = \frac{1 - f(x)}{1 + f(x)},$$

then

$$g'(x) = \frac{-f'(x)(1+f(x)) - (1-f(x))f'(x)}{(1+f(x))^2} = \frac{-2f'(x)}{(1+f(x))^2},$$

which is defined and positive on $\mathbb{R}$, since $f(x) > 0 > -1$ and $f'(x) < 0$ for all real values of x. Therefore, the function g is increasing on $\mathbb{R}$.

12. If x is the length of the side of the base of the box and y is the height of the box (both measured in metres), then $x^2y = 2$, *i.e.*, $y = 2x^{-2}$, the side of the original square sheet is $x + 2y$ metres long, and it is required to minimize the area $A = (x+2y)^2 = \bigl(x + 4x^{-2}\bigr)^2$, where $x > 0$. The derivative,

$$\frac{dA}{dx} = 2\bigl(x + 4x^{-2}\bigr)\bigl(1 - 8x^{-3}\bigr),$$

is negative if $0 < x < 2$, zero if $x = 2$ and positive if $x > 2$, so the area of the original square is minimized if $x = 2$, in which case $y = \frac{1}{2}$. Therefore, the area of the original square sheet is minimized if the base of the box has side length two metres and the box is one-half of a metre high.

13. If $f''(x) = -\sin x$, then $f(x) = \sin x + Ax + B$. where $A, B \in \mathbb{R}$. Then $B = f(0) = -1$, and $\pi = f\bigl(\frac{1}{2}\pi\bigr) = \frac{1}{2}\pi A$, so $A = 2$. Therefore, $f(x) = \sin x + 2x - 1$.

14. a. Termwise integration gives

$$\int (x^5 + 5^x + \log 5)\,dx = \tfrac{1}{6}x^6 + 5^x/\log 5 + x\log 5 + C.$$

b. Expanding the integrand and then integrating termwise gives

$$\begin{aligned} \int_1^e \left(1 - \frac{1}{x}\right)^2 dx &= \int_1^e \bigl(1 - 2x^{-1} + x^{-2}\bigr)\,dx \\ &= \bigl(x - 2\log(x) - x^{-1}\bigr)\Big|_1^e \\ &= e - 2 - e^{-1}. \end{aligned}$$

c. Expanding the integrand and then integrating termwise gives

$$\begin{aligned} \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \tan x(\cos x - \sec x)\,dx &= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} (\sin x - \sec x\tan x)\,dx \\ &= -(\cos x + \sec x)\Big|_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \\ &= -\bigl(\tfrac{1}{2}\sqrt{2} + \sqrt{2}\bigr) + \bigl(\tfrac{1}{2}\sqrt{3} + \tfrac{2}{3}\sqrt{3}\bigr) \\ &= \tfrac{7}{6}\sqrt{3} - \tfrac{3}{2}\sqrt{2}. \end{aligned}$$

d. Revising the integrand and then integrating termwise gives

$$\begin{aligned} \int \left(\sqrt{2x} + 2x\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx &= \int \bigl(2x^{3/2} + \sqrt{2}\,x^{1/2} + x^{-1/2}\bigr)\,dx \\ &= \tfrac{4}{5}x^{5/2} + \tfrac{2}{3}\sqrt{2}\,x^{3/2} + 2x^{1/2} + C. \end{aligned}$$

15. From the given description,

$$g(x) = \int_0^x (1 + \sin^2 t)\,dt,$$

and so

$$g'(x) = 1 + \sin^2 x$$

by the (first form of the) Fundamental Theorem of Calculus.

16. If the interval $[\,0,1\,]$ is divided into n subintervals of equal length then the length of each subinterval is $1/n$ and the endpoints of the subintervals are $x_i = i/n$, for $0 \leqslant i \leqslant n$. The Riemann sum obtained by evaluating $4x - x^2$ at the right endpoints of the subintervals is

$$\begin{aligned} \mathscr{R}_n = \frac{1}{n}\sum_{i=1}^n \left\{4\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^2\right\} &= \frac{4}{n^2}\sum_{i=1}^n i - \frac{1}{n^3}\sum_{i=1}^n i^2 \\ &= \frac{4}{n^2}\cdot\frac{1}{2}n(n+1) - \frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1) \\ &= 2\left(1 + \frac{1}{n}\right) - \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right). \end{aligned}$$

Therefore,

$$\int_0^1 (4x - x^2)\,dx = \lim_{n\to\infty} \mathscr{R}_n = 2 - \tfrac{1}{6}\cdot 2 = \tfrac{5}{3}.$$