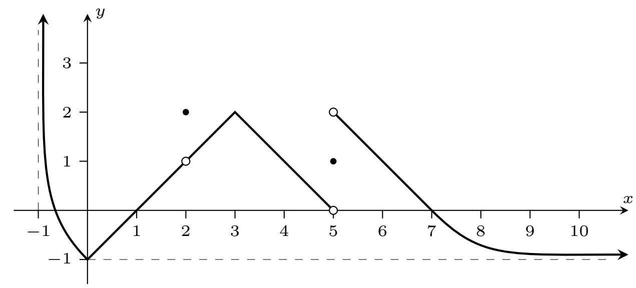
1. By referring to the graph below, evaluate the following expressions.



a. $\lim_{x\to 5^-} f(x)$ b. $\lim_{x\to 5^+} f(x)$ c. $f\big(f(5)\big)$

d. $\lim_{x\to -1^+} f(x)$ e. $\lim_{h\to 0} \dfrac{f\left(\frac{5}{4}+h\right)-f\left(\frac{5}{4}\right)}{h}$ f. $f''(4)$

2. Evaluate each of the following limits.

a. $\lim_{x\to 1} \dfrac{3x^2-x-2}{x^2-x}$ b. $\lim_{x\to\infty} \left(\dfrac{1+x}{6+5x^2}\right)\left(\dfrac{5+7x^3}{2-5x^2}\right)$ c. $\lim_{\vartheta\to 0} \dfrac{\tan(6\vartheta)}{\sin(8\vartheta)}$

d. $\lim_{x\to\infty} \left\{\sqrt{4x^2+3x-2}-3x\right\}$ e. $\lim_{x\to 1^+} \dfrac{x+1}{x-|2-3x|}$

3. Let

$$f(x)=\begin{cases}\dfrac{1}{k+1-x} & \text{if } x\leqslant 3, \text{ and}\\[2ex] \sqrt{\dfrac{x^2-5x+6}{k(x-3)}} & \text{if } x>3.\end{cases}$$

Find all values of k which make the function f continuous at 3.

4. Find an equation of the normal line to the curve defined by

$$y=\frac{x^2}{x-2}$$

at the point whose x-coordinate is 3.

5. Use the limit definition of the derivative to find $f'(x)$, where

$$f(x)=\frac{1}{2x+1}.$$

6. Find $\dfrac{dy}{dx}$ for each of the following.

a. $y=\dfrac{\sqrt[3]{x}}{2}+\dfrac{2}{x+1}-3^x+\cos(e^2)$ b. $y=\dfrac{x}{x+1}+\log\left(\dfrac{2}{x}\right)$

c. $y=\csc^2(3x^2)+\log(4-x)+xe^{3x^2}$ d. $y=(x-1)(2+x)^{2x}$

e. $\sin(x-y)=xy$

7. A particle moves along a straight line with its position at time t given by $s=t^{2/3}(20-t)$. What is the distance travelled by the particle during the time interval $[1,27]$?

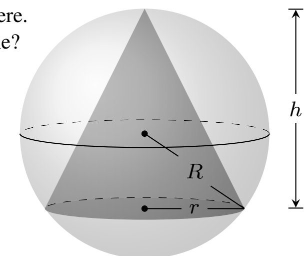
8. Show that the function f, defined by $f(x)=e^x(x^3-3x^2+6x+2)$, has exactly one real zero.

9. Find all points on the graph of $x^4+y^4+2=4xy^3$ at which the tangent line is horizontal.

10. A plane, flying in a straight line at a constant altitude of 4 km, passes directly over a telescope tracking it. At a certain moment the angle between the telescope's line of sight and the ground is $\frac{1}{3}\pi$ and is decreasing at a rate of $\frac{1}{2}$ radians per minute. How fast is the plane travelling at that moment?

11. Find the absolute extrema of $f(x)=15+12x-x^3$ on the interval $[1,4]$.

12. A right circular cone is inscribed in a sphere. What is the largest possible volume of the cone?



13. Sketch the graph of f, where

$$f(x)=\frac{6-2e^x}{1+e^x},\quad f'(x)=\frac{-8e^x}{(1+e^x)^2}\quad\text{and}\quad f''(x)=\frac{8e^x(e^x-1)}{(1+e^x)^3}.$$

Include all intercepts, asymptotes, intervals of monotonicity and concavity, extreme values and points of inflection.

14. Given that $f'(x)=3\sin x+\pi^{-1}$ and $f\left(\frac{3}{4}\pi\right)=0$, find $f(x)$.

15. Compute the definite integral

$$\int_1^4 (x^2-x+1)\,dx$$

as a limit of Riemann sums.

16. Find a number b and a function f such that

$$\int_2^b f(x)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{3+\frac{4i}{n}}\left(\frac{4}{n}\right).$$

17. Evaluate the definite integral

$$\int_1^2 \left(|x|+\sqrt{4-x^2}\,\right)dx$$

by interpreting it in terms of area.

18. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{(\sqrt{x}-1)^2}{x}\,dx$ b. $\displaystyle\int \left(e^{x+1}+\tfrac{1}{2}\sec x\tan x+3\sec^2 x\right)dx$

c. $\displaystyle\int_1^e \left(\frac{2}{t}+\frac{1}{e}\right)dt$

19. Find the numbers in $(0,2\pi)$ at which F has local extrema, where

$$F(x)=\int_x^{2x}\frac{\sin t}{t}\,dt.$$

1. By inspecting the graph of f,
a. $\lim_{x\to 5^-} f(x)=0$, b. $\lim_{x\to 5^+} f(x)=2$, c. $f(f(5))=f(1)=0$,

d. $\lim_{x\to -1^+} f(x)=\infty$, e. $\lim_{h\to 0}\dfrac{f\big(\frac{5}{4}+h\big)-f\big(\frac{5}{4}\big)}{h}=1$, f. $f''(4)=0$.

2. a. The denominator vanishes as $x\to 1$ so the limit cannot be evaluated by direct substitution alone. But factorizing the numerator and denominator of the expression in the limit gives

$$\lim_{x\to 1}\frac{(3x+2)(x-1)}{x(x-1)}=\lim_{x\to 1}\frac{3x+2}{x}=\frac{3(1)+2}{1}=5,$$

by independence and direct substitution.
b. Extracting the dominant powers from the numerator and the denominator of the expression in the limit gives

$$\lim_{x\to\infty}\left\{\frac{1+1/x}{6/x^2+5}\cdot\frac{5/x^2+7}{2/x^2-5}\right\}=\frac{1\cdot 7}{5\cdot(-5)}=-\tfrac{7}{25},$$

by arithmetical properties of limits, and the fact that $1/x^n\to 0$ as $x\to\infty$ for any positive integer n.
c. Revising the expression in the limit gives

$$\lim_{\vartheta\to 0}\frac{\tan(6\vartheta)}{\sin(8\vartheta)}=\lim_{\vartheta\to 0}\left\{\frac{\sin(6\vartheta)}{6\vartheta}\cdot\frac{1}{\cos(6\vartheta)}\cdot\frac{8\vartheta}{\sin(8\vartheta)}\cdot\frac{6}{8}\right\}=\tfrac{3}{4},$$

by arithmetical properties of limits and the fact that $(\sin t)/t\to 1$ as $t\to 0$.
d. Extracting dominant powers gives

$$\lim_{x\to\infty}\left\{x\big(\sqrt{4+3/x-2/x^2}-3\big)\right\}=-\infty,$$

since the limit as $x\to\infty$ of the right factor is $-1<0$ (because $1/x^n\to 0$ as $x\to\infty$ for any positive integer n).
e. If $x>1$ then $x-|2-3x|=x+(2-3x)=2(1-x)$, which implies that

$$\lim_{x\to 1^+}\frac{x+1}{x-|2-3x|}=\lim_{x\to 1^+}\frac{x+1}{2(1-x)}=-\infty,$$

since $x+1\to 2$, and $2(1-x)\to 0^-$, as $x\to 1^+$.

3. Since

$$f(3)=\lim_{x\to 3-}f(x)=\frac{1}{k-2},\text{ and }\lim_{x\to 3^+}f(x)=\lim_{x\to 3^+}\sqrt{\frac{x-2}{k}}=\frac{1}{\sqrt{k}},$$

by independence and direct substitution, the function f is continuous at 3 if, and only if $k>2$ and $(k-2)^2=k$, *i.e.*, $k=4$.

4. If $y=x^2/(x-2)$ and $x=3$, then $y=3^2/(3-2)=9$, and

$$\left.\frac{dy}{dx}\right|_{x=3}=\left.\frac{x(x-4)}{(x-2)^2}\right|_{x=3}=\frac{3(3-4)}{(3-2)^2}=-3.$$

The normal line to the curve at the point $(3,9)$, since it is perpendicular to the tangent line, has slope $\frac{1}{3}$, and equation $y-9=\frac{1}{3}(x-3)$, or $x-3y=-24$.

5. If $f(x)=1/(2x+1)$, then $f'(x)$ is equal to

$$\lim_{t\to x}\frac{\dfrac{1}{2t+1}-\dfrac{1}{2x+1}}{t-x}=\lim_{t\to x}\frac{2(x-t)}{(2t+1)(2x+1)(t-x)}=\frac{-2}{(2x+1)^2},$$

by independence and direct substitution.

6. a. If $y=\frac{1}{2}\sqrt[3]{x}+2/(x+1)-3^x+\cos(e^2)$, then

$$\frac{dy}{dx}=\tfrac{1}{6}x^{-2/3}-\frac{2}{(x+1)^2}-3^x\log 3.$$

b. If $y=x/(x+1)+\log(2/x)=1-1/(x+1)+\log 2-\log x$, then

$$\frac{dy}{dx}=\frac{1}{(x+1)^2}-\frac{1}{x}.$$

c. If $y=\csc^2(3x^2)+\log(4-x)+xe^{3x^2}=1/\sin^2(3x^2)+\log(4-x)+xe^{3x^2}$, then

$$\frac{dy}{dx}=\frac{-12x\cos(3x^2)}{\sin^3(3x^2)}+\frac{1}{x-4}+e^{3x^2}(1+6x^2).$$

d. If $y=(x-1)(2+x)^{2x}$, then logarithmic differentiation gives

$$\begin{aligned}\frac{dy}{dx}&=(x-1)(2+x)^{2x}\frac{d}{dx}\{\log(x-1)+2x\log(2+x)\}\\&=(x-1)(2+x)^{2x}\{1/(x-1)+2\log(2+x)+2x/(2+x)\}\\&=(2+x)^{2x-1}\{2+x+2(x-1)\big(x+(2+x)\log(2+x)\big)\}\end{aligned}$$

e. If $\sin(x-y)=xy$, then differentiating implicitly with respect to x gives

$$\cos(x-y)\left(1-\frac{dy}{dx}\right)=y+x\frac{dy}{dx},\qquad\text{and so}\qquad\frac{dy}{dx}=-\frac{y-\cos(x-y)}{x+\cos(x-y)}.$$

7. If the position of the particle is given by $s=t^{2/3}(20-t)=20t^{2/3}-t^{5/3}$, then its velocity,

$$\frac{ds}{dt}=\tfrac{40}{3}t^{-1/3}-\tfrac{5}{3}t^{2/3}=\frac{5(8-t)}{3t^{2/3}},$$

is positive if $0<t<8$ (when the particle is moving in the positive direction), zero if $t=8$ (when the particle is at rest) and negative if $t>8$ (when the particle is moving in the negative direction). So the distance travelled by the particle during the time interval $[1,27]$ is equal to $s(8)-s(1)+s(8)-s(27)$, or

$$2\cdot 8^{2/3}(12)-1^{2/3}(19)-27^{2/3}(-7)=96-19+63=140.$$

8. If $f(x)=e^x(x^3-3x^2+6x+2)$ and $g(x)=e^{-x}f(x)$, then g has the same zeros as f, and is differentiable (and therefore also continuous) on $\mathbb{R}$, because it is a polynomial function. Since $g(-1)=-8$ and $g(0)=2$, the Intermediate Value Theorem implies that there is a real number α such that $-1<\alpha<0$ and $g(\alpha)=0$. On the other hand, if $\alpha\neq\beta$, then the Mean Value Theorem gives a real number μ between α and β such that

$$|g(\beta)|=|g(\alpha)+g'(\mu)(\beta-\alpha)|\geqslant 3|\beta-\alpha|\neq 0,$$

since $g'(x)=3x^2-6x+6=3(x-1)^2+3\geqslant 3$, for all real values of x. So α is the only real zero of g, and therefore also the only real zero of f.

9. If $x^4+y^4+2=4xy^3$, then differentiating implicitly with respect to x gives

$$4x^3+4y^3\frac{dy}{dx}=4y^3+12xy^2\frac{dy}{dx},\qquad\text{and so}\qquad\frac{dy}{dx}=-\frac{x^3-y^3}{y^2(y-3x)},$$

which is equal zero if, and only if, $x=y$ and $y\neq 0$. Replacing y by x in the equation of the curve gives $2x^4+2=4x^4$, or $x^4=1$, whose solutions are ± 1. Therefore, the tangent line to the given curve is horizontal at the points $(\pm 1,\pm 1)$.

10. If ϑ denotes the angle between the telescope's line of sight and the vertical, and x denotes the distance (measured in kilometres) between the telescope and the point on the ground directly below the plane, then $x=4\tan\vartheta$, and so

$$\frac{dx}{dt}=(4\sec^2\vartheta)\frac{d\vartheta}{dt}.$$

At the moment in question, $\sec^2\vartheta=\sec^2\big(\frac{1}{2}\pi-\frac{1}{3}\pi\big)=\frac{4}{3}$, $d\vartheta/dt=\frac{1}{2}$, so the plane is moving at $4\cdot\frac{4}{3}\cdot\frac{1}{2}=\frac{8}{3}$ kilometres per minute, or 160 kilometres per hour.

11. If $f(x)=15+12x-x^3$, then $f'(x)=12-3x^2=3(2-x)(2+x)$, and so ± 2 are the critical numbers of f, of which only 2 lies in $(1,4)$. Since

$$f(1)=26,\qquad f(2)=31\qquad\text{and}\qquad f(4)=-1,$$

the largest and smallest values of f on $[1,4]$ are, respectively, 31 and -1.

12. If h denotes the height of the inscribed cone and r denotes the radius of its base, then the volume of the cone is $V=\frac{1}{3}\pi r^2h$. Since the cone is inscribed in a sphere of radius R, Pythagoras' formula implies that $r^2+(h-R)^2=R^2$, or $r^2=R^2-(h-R)^2=h(2R-h)$, and so the volume of the cone is equal to

$$V=\tfrac{1}{3}\pi h^2(2R-h)=\tfrac{1}{3}\pi(2Rh^2-h^3),$$

where $0\leqslant h\leqslant 2R$. Since

$$\frac{dV}{dh}=\tfrac{1}{3}\pi(4Rh-3h^2)=\tfrac{1}{3}\pi h(4R-3h),$$

the only critical number of V in $(0,2R)$ is $\frac{4}{3}R$, and since $V=0$ if h is 0 or $2R$, it follows that the largest volume of a right circular cone inscribed in a sphere of radius R is equal to

$$\left.V\right|_{h=\frac{4}{3}R}=\tfrac{1}{3}\pi\big(\tfrac{4}{3}R\big)^2\big(2R-\tfrac{4}{3}R\big)=\tfrac{32}{81}\pi R^3,$$

which is $\frac{8}{27}$ of the volume of the sphere.

13. The domain of f is the set of all real numbers, on which f is differentiable (and therefore also continuous), so the graph of f has no vertical asymptotes. The intercepts of the graph of f are $(0, 2)$ and $(\log 3, 0)$, where the latter is found by solving $0 = 6 - 2e^x = 2(3 - e^x)$. Since

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} \frac{2(3-e^x)}{e^x+1} = 6,$$

and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2(3e^{-x}-1)}{1+e^{-x}} = -2,$$

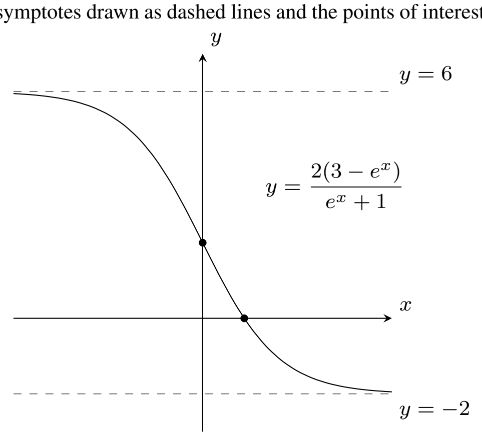
the horizontal asymptotes of the graph of f are defined by $y = 6$ (as $x \to -\infty$) and $y = -2$ (as $x \to \infty$). Next, because

$$f'(x) = \frac{-8e^x}{(e^x+1)^2}$$

is negative for all real values of x, it follows that f is decreasing on $\mathbb{R}$ and has no local or global extreme values. Finally,

$$f''(x) = \frac{8e^x(e^x-1)}{(e^x+1)^3}$$

so the graph of f is concave down where $x < 0$, concave up where $x > 0$ and has a point of inflection at its y intercept. Below is a sketch of the graph of f, with the horizontal asymptotes drawn as dashed lines and the points of interest emphasized.



14. If $f'(x) = 3\sin x + \pi^{-1}$ and $f\left(\frac{3}{4}\pi\right) = 0$, then

$$f(x) = \int_{\frac{3}{4}\pi}^{x} \left(3\sin t + \pi^{-1}\right) dt = -3\cos(x) + \pi^{-1}x - \tfrac{3}{2}\sqrt{2} - \tfrac{3}{4}.$$

15. If $[\,1, 4\,]$ is divided into n subintervals of equal length then the length of each subinterval is $\Delta x = 3/n$. The endpoints of the subintervals are $x_i = 1 + 3i/n$, and the values of f at these endpoints are

$$f(x_i) = \left(1 + \frac{3}{n}i\right)^2 - \left(1 + \frac{3}{n}i\right) + 1 = 1 + \frac{3}{n}i + \frac{9}{n^2}i^2,$$

where $i = 0, 1, 2, \ldots, n$. Evaluating f at the right endpoint of each subinterval gives the Riemann sum

$$\begin{aligned}\mathscr{R}_n &= \frac{3}{n}\sum_{i=1}^{n}\left\{1 + \frac{3}{n}i + \frac{9}{n^2}i^2\right\} = \frac{3}{n}\sum_{i=1}^{n}1 + \frac{9}{n^2}\sum_{i=1}^{n}i + \frac{27}{n^3}\sum_{i=1}^{n}i^2\\ &= \frac{3}{n}\cdot n + \frac{9}{n^2}\cdot\frac{1}{2}n(n+1) + \frac{27}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)\\ &= 3 + \frac{9}{2}\left(1 + \frac{1}{n}\right) + \frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right),\end{aligned}$$

and so

$$\int_1^4 (x^2 - x + 1)\,dx = \lim_{n\to\infty}\mathscr{R}_n = 3 + \tfrac{9}{2} + 9 = \tfrac{33}{2}.$$

16. If $f(x) = \sqrt{\alpha x + \beta}$ and $b > 2$, and the interval $[\,2, b\,]$ is divided into n subintervals of equal length, then the length of each subinterval is $(b-2)/n$, the endpoints of the subintervals are $x_i = 2 + (b-2)i/n$, and the values of f at these endpoints are

$$f(x_i) = \sqrt{\alpha\left(2 + \frac{b-2}{n}i\right) + \beta} = \sqrt{2\alpha + \beta + \frac{\alpha(b-2)}{n}i},$$

for $i = 0, 1, 2, \ldots, n$. The given limit is equal to the definite integral of f on $[\,2, b\,]$ provided (among other possibilities)

$$\frac{b-2}{n}\sqrt{2\alpha + \beta + \frac{\alpha(b-2)}{n}i} = \frac{4}{n}\sqrt{3 + \frac{4}{n}i},$$

or (since no factor is ever negative)

$$\frac{(b-2)^2(2\alpha+\beta)}{n^2} + \frac{\alpha(b-2)^3}{n^3}i = \frac{48}{n^2} + \frac{64}{n^3}i,$$

for $i = 1, 2, 3, \ldots, n$. If $n \geqslant 2$, these last equations are equivalent to the equations

$$(b-2)^2(2\alpha+\beta) = 48 \qquad \text{and} \qquad \alpha(b-2)^3 = 64.$$

Plainly there are infinitely many pairs f, b which satisfy the given equation. Here are three such pairs: Taking $b = 6$, $\alpha = 1$ and $\beta = 1$ gives

$$\int_2^6 \sqrt{x+1}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\sqrt{3 + \frac{4}{n}i};$$

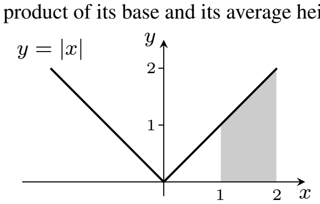
taking $b = 3$, $\alpha = 64$ and $\beta = -80$ gives

$$\int_2^3 \sqrt{64x - 80}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\sqrt{3 + \frac{4}{n}i};$$

taking $b = \frac{14}{3}$, $\alpha = \frac{27}{8}$ and $\beta = 0$ gives

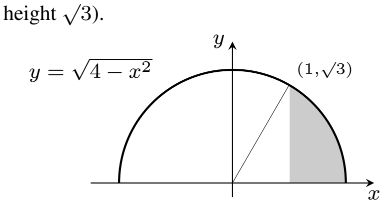
$$\int_2^{\frac{14}{3}} \sqrt{\tfrac{27}{8}x}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\sqrt{3 + \frac{4}{n}i}.$$

17. The integral below represents the area of the shaded trapezoid, which is the product of its base and its average height.



$$\int_1^2 |x|\,dx = \tfrac{1}{2}(1+2)(2-1) = \tfrac{3}{2}.$$

The integral below represents the area of the shades circular segment, which is the difference of a sector (of radius 2 and angle $\frac{1}{3}\pi$) and a triangle (of base 1 and height $\sqrt{3}$).



$$\int_1^2 \sqrt{4-x^2}\,dx = \tfrac{2}{3}\pi - \tfrac{1}{2}\sqrt{3}.$$

Therefore,

$$\int_1^2 \left(|x| + \sqrt{4-x^2}\,\right)dx = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{3} + \tfrac{2}{3}\pi.$$

18. a. Expanding the integrand and integrating termwise by inspection gives

$$\int \frac{(\sqrt{x}-1)^2}{x}\,dx = \int\left\{1 - \frac{2}{\sqrt{x}} + \frac{1}{x}\right\}dx = x - 4\sqrt{x} + \log|x| + C.$$

b. Integrating termwise by inspection gives

$$\int\left\{e^{x+1} + \tfrac{1}{2}\sec x\tan x + 3\sec^2 x\right\}dx = e^{x+1} + \tfrac{1}{2}\sec x + 3\tan x + C.$$

c. Integrating termwise by inspection gives

$$\int_1^3 \left\{\frac{2}{t} + \frac{1}{e}\right\}dt = \left\{2\log t + \frac{t}{e}\right\}\Bigg|_1^e = 3 - e^{-1}.$$

19. Since

$$F(x) = \int_x^{2x}\frac{\sin t}{t}\,dt = \int_1^{2x}\frac{\sin t}{t}\,dt - \int_1^{x}\frac{\sin t}{t}\,dt,$$

the Chain Rule and the Fundamental Theorem of Calculus imply that

$$F'(x) = \frac{\sin 2x}{2x}\cdot 2 - \frac{\sin x}{x} = \frac{\sin x}{x}(2\cos x - 1),$$

which is positive on $\left(0, \frac{1}{3}\pi\right)$ and on $\left(\pi, \frac{5}{3}\pi\right)$, where F is increasing, and is negative on $\left(\frac{1}{3}\pi, \pi\right)$ and on $\left(\frac{5}{3}\pi, 2\pi\right)$, where F is decreasing. So F has local maxima at $\frac{1}{3}\pi$ and at $\frac{5}{3}\pi$, and a local minimum at π.