1. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 2} \frac{\dfrac{1}{x+4} - \dfrac{1}{3x}}{x-2}$

b. $\displaystyle\lim_{x\to \frac{1}{3}\pi^+} \frac{\sqrt{x}}{2\cos x - 1}$

c. $\displaystyle\lim_{x\to -\infty} \frac{9-3x}{2x - \sqrt{4x^2+3x-9}}$

d. $\displaystyle\lim_{x\to \frac{1}{3}} \frac{9x^2-1}{8x-3|x^2-1|}$

e. $\displaystyle\lim_{x\to 0^+} e^{-5/x}\sin(5/x)$

2. Find all values of a and b so that the function f, defined by

$$f(x) = \begin{cases} x^2+6x-18 & \text{if } x < b, \\ a & \text{if } x = b \text{ and} \\ 5x-6 & \text{if } x > b, \end{cases}$$

is continuous everywhere.

3. Sketch the graph of a function f such that

- the graph of f has horizontal asymptotes defined by $y=-2$ and $y=2$,
- $\displaystyle\lim_{x\to 3} f(x)$ exists, but f is not continuous at 3,
- f is continuous but not differentiable at -1.

4. Consider the function f, defined by

$$f(x) = \frac{2}{\sqrt{4-3x}}.$$

a. Find $f'(x)$ using the limit definition of the derivative.

b. Find $f'(x)$ using the rules of differentiation, and check that your answer matches with Part a.

c. Find an equation of the tangent to the graph of f at the point where $x=1$.

5. Find $\dfrac{dy}{dx}$ for each of the following.

a. $y = x(x+3)^5\log x$

b. $y = \dfrac{(e^x+x^4)^3}{1+\sec 5x}$

c. $\tan(x+y) = x+y^2$

d. $x^y = y^{\sin x}$

6. Use logarithmic differentiation to find the derivative with respect to x of

$$y = \frac{\sec^3(x)\sqrt[3]{x+2}}{x^x(2x+1)^7}.$$

7. Find the interval(s) on which the graph of $y = xe^{3x}$ is concave down.

8. A sample of gas is placed in a cylindrical chamber, held at constant temperature, and a piston is seated on top, The piston is lowered at a constant rate of 3 cm/s. When the bottom of the piston is 6 cm from the base of the chamber, the pressure in the chamber is 100 kPa. Find the rate of change of the pressure in the chamber at this instant. (You are given Boyle's Law: pv is constant, where p is the pressure of the gas and v is the volume of the gas.)

9. Show that the equation $e^x = 3x$ has exactly two real solutions.

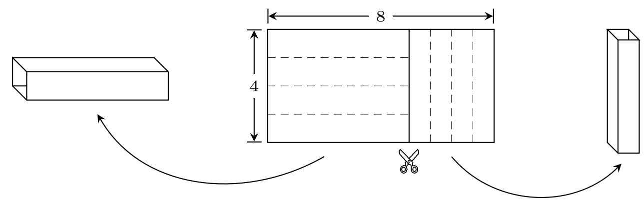
10. Find the global extrema of $f(x) = x - 6\sqrt[3]{2x-3}$ on the interval $[-12, 15]$.

11. Sketch the graph of

$$f(x) = \frac{(x-1)^2(2x+1)}{x^3} = 2 - \frac{3}{x} + \frac{1}{x^3}.$$

Be sure that your solution includes the domain of f, and all intercepts, asymptotes, intervals of monotonicity and concavity, local extrema and points of inflection.

12. A rectangular piece of paper measuring 4 inches by 8 inches is cut across its long side into two rectangles (either of which may be degenerate). Each piece is then folded along the dashed lines, making square prisms with open ends as in the figure below.



Where should the cutting line be so that the total volume of the square prisms is

a. minimized?

b. maximized?

13. Find $f(t)$ if $f''(t) = 3\sqrt{t}+2$, $f(1)=0$ and $f'(1)=3$.

14. Use differentiation to verify that

$$\int \cos^3 x\,dx = \tfrac{1}{3}(2+\cos^2 x)\sin x + C.$$

15. Express the definite integral

$$\int_1^3 \frac{\log x}{x}\,dx$$

as a limit of Riemann sums.

16. Evaluate each of the following integrals.

a. $\displaystyle\int \left\{\sqrt{\frac{6}{x}} + \sqrt{6}\,x + 6^x + e^6\right\} dx$

b. $\displaystyle\int_1^2 \frac{x^2+4x+3}{x^2+x}\,dx$

c. $\displaystyle\int (\tan x + \sec^3 x)\cos x\,dx$

d. $\displaystyle\int_0^5 \big(|x-2|+1\big)\,dx$

17. Find the derivative with respect to x of

$$F(x) = \int_2^{e^x} \frac{t}{\log t}\,dt.$$

18. Find the polynomial $p(x)$ such that

$$\lim_{x\to\infty} \frac{p(x)}{x^2-x-6} = 2 \quad \text{and} \quad \lim_{x\to 3} \frac{p(x)}{x^2-x-6} = 1.$$

1. a. Simplifying the expression in the limit gives

$$\frac{\dfrac{1}{x+4}-\dfrac{1}{3x}}{x-2}=\frac{2(x-2)}{3x(x+4)(x-2)}=\frac{2}{3x(x+4)},$$

at least if $x \neq 2$, so the limit in question is equal to

$$\lim_{x\to 2}\frac{2}{3x(x+4)}=\frac{2}{3\cdot 2(2+4)}=\tfrac{1}{18},$$

by independence and direct substitution.

b. As $\sqrt{x}\to\sqrt{\frac{1}{3}\pi}>0$ and $2\cos(x)-1\to 0^-$, as $x\to\frac{1}{3}\pi^+$, it follows that

$$\lim_{x\to\frac{1}{3}\pi^+}\frac{\sqrt{x}}{2\cos x-1}=-\infty.$$

c. If $t=1/x$, then $\sqrt{t^2}=-t$ since $t<0$, and

$$\lim_{x\to-\infty}\frac{9-3x}{2x-\sqrt{4x^2+3x-9}}=\lim_{t\to 0^-}\frac{9t-3}{2+\sqrt{4+3t-9t^2}}=-\tfrac{3}{4}.$$

d. If $|x|<1$ then $8-3|x^2-1|=3x^2+8x-3=(3x-1)(x+3)$, so the limit in question is equal to

$$\lim_{x\to\frac{1}{3}}\frac{(3x-1)(3x+1)}{(3x-1)(x+3)}=\lim_{x\to\frac{1}{3}}\frac{3x+1}{x+3}=\frac{1+1}{\frac{1}{3}+3}=\tfrac{3}{5},$$

by independence and direct substitution.

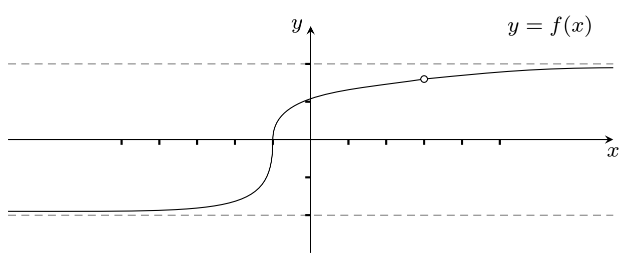
e. If $0<\varepsilon<1$ and $0<x<-5/\log(\varepsilon)$, then $|e^{-5/x}\sin(5/x)|\leqslant e^{-5/x}<\varepsilon$, so $\lim\limits_{x\to 0^+}e^{-5/x}\sin(5/x)=0$ by definition.

2. Since the parts of f are defined by polynomial functions, f is continuous at all real numbers, except possibly b, where

$$\lim_{x\to b^-}f(x)=b^2+6b-18,\quad f(b)=a\quad\text{and}\quad\lim_{x\to b^+}f(x)=5b-6.$$

So f is continuous at b, and hence on $\mathbb{R}$, if, and only if, $b^2+6b-18=5b-6$ and $a=5b-6$. The first equation is equivalent to $0=b^2+b-12=(b+4)(b-3)$, *i.e.*, $b=-4$ or $b=3$, and the second equation gives, respectively, $a=-26$ and $a=9$.

3. Consider the graph below (with unit lengths marked along the coordinate axes).

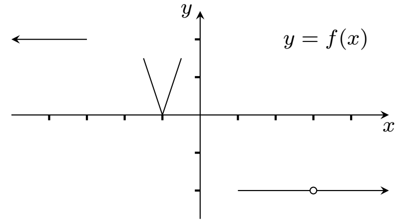


The dashed lines indicate that $\lim\limits_{x\to\pm\infty}f(x)=\pm 2$, so $y=-2$ and $y=2$ define the horizontal asymptotes of the graph of f. Also, $\lim\limits_{x\to 3}f(x)$ exists but 3 is not in the domain of f, so f is not continuous at 3. Finally, the graph is intended to suggest that f is continuous at -1 but

$$\lim_{t\to -1}\frac{f(t)-f(-1)}{t+1}=\infty,$$

which implies that f is not differentiable at -1.

The domain of f was not specified, so the sketch below is correct and simpler.



4. a. If $f(x)=2/\sqrt{4-3x}=2(4-3x)^{-1/2}$, then

$$\begin{aligned}f(t)-f(x)&=\frac{2}{\sqrt{4-3t}}-\frac{2}{\sqrt{4-3x}}\\&=2\,\frac{\sqrt{4-3x}-\sqrt{4-3t}}{\sqrt{4-3t}\sqrt{4-3x}}\cdot\frac{\sqrt{4-3x}+\sqrt{4-3t}}{\sqrt{4-3x}+\sqrt{4-3t}}\\&=\frac{6(t-x)}{\big(\sqrt{4-3x}+\sqrt{4-3t}\big)\sqrt{4-3t}\sqrt{4-3x}}.\end{aligned}$$

Therefore, by the definition of the derivative,

$$\begin{aligned}f'(x)&=\lim_{t\to x}\frac{6}{\big(\sqrt{4-3x}+\sqrt{4-3t}\big)\sqrt{4-3t}\sqrt{4-3x}}\\&=\frac{3}{(4-3x)\sqrt{4-3x}}=3(4-3x)^{-3/2}.\end{aligned}$$

b. The Power Rule, the Chain Rule, and linearity of the derivative, give

$$f'(x)=2\big(-\tfrac{1}{2}\big)(4-3x)^{-3/2}(-3)=3(4-3x)^{-3/2}.$$

c. Since $f(1)=2$ and $f'(1)=3$, the tangent line to the graph of f at the point $(1,2)$ is defined by $y=2+3(x-1)$, or $3x-y=1$.

5. a. If $y=x(x+3)^5\log x$, then

$$\begin{aligned}\frac{dy}{dx}&=(x+3)^5\log x+5x(x+3)^4\log x+(x+3)^5\\&=(x+3)^4\big(x+3+3(2x+1)\log x\big).\end{aligned}$$

b. If $y=(e^x+x^4)^3/(1+\sec 5x)$, then

$$\frac{dy}{dx}=\frac{3(e^x+x^4)^2(e^x+4x^3)}{1+\sec 5x}-\frac{5(e^x+x^4)\sec 5x\tan 5x}{(1+\sec 5x)^2}.$$

c. If $\tan(x+y)=x+y^2$, *i.e.*, $\tan(x+y)-x-y^2=0$, then

$$\frac{dy}{dx}=-\frac{\sec^2(x+y)-1}{\sec^2(x+y)-2y}.$$

d. The equation $x^y=y^{\sin x}$ is equivalent to the equation $y\log x=\sin x\log y$, or $y\log x-\sin x\log y=0$, and so

$$\frac{dy}{dx}=-\frac{y/x-\cos x\log y}{\log x-(\sin x)/y}=-\frac{y(y-x\cos x\log y)}{x(y\log x-\sin x)}.$$

6. If

$$y=\frac{\sec^3(x)\sqrt[3]{x+2}}{x^x(2x+1)^7},$$

then logarithmic differentiation gives

$$\begin{aligned}\frac{dy}{dx}&=y\frac{d}{dx}\big\{\log|y|\big\}\\&=\frac{\sec^3(x)\sqrt[3]{x+2}}{x^x(2x+1)^7}\left\{3\tan x+\frac{1}{3(x+2)}-\log x-1-\frac{14}{2x+1}\right\}.\end{aligned}$$

7. If $f(x)=xe^{3x}$, then $f''(x)=e^{3x}\big(3(1+3x)+3\big)=3e^{3x}(3x+2)$, which is negative if, and only if, $3x+2<0$, *i.e.*, $x<-\frac{2}{3}$. Therefore, the graph of f is concave down on the interval $\big(-\infty,-\frac{2}{3}\big)$.

8. Since all cross sections of the cylinder parallel to the base have the same area, Boyle's Law implies that the product of the pressure p in the chamber (measured in kPa) and the distance h between the bottom of the piston and the base of the chamber (measured in cm) is constant. It follows that

$$\frac{dp}{dt}h+p\frac{dh}{dt}=0,\qquad\text{or}\qquad\frac{dp}{dt}=-\frac{p}{h}\frac{dh}{dt}=3\frac{p}{h},$$

since it is given that the piston is being lowered at a constant rate of 3 centimetres per second. When $h=6$ and $p=100$, $\frac{dp}{dt}=3\cdot\frac{100}{6}=50$, *i.e.*, the pressure in the chamber is increasing at a rate of $50\,\mathrm{kPa/s}$ at the instant in question.

9. Let $f(x)=e^x-3x$; the real zeros of f are the real solutions of the equation in question, so it suffices to prove that f has exactly two real zeros. f is differentiable on $\mathbb{R}$, so the Intermediate Value Theorem and Rolle's Theorem apply to f on any closed interval of positive length. Since $f(0)=1>0$, $f(1)=e-3<0$ and

$$f(2)=e^2-6>\big(\tfrac{5}{2}\big)^2-6=\tfrac{25}{4}-6=\tfrac{1}{4}>0,$$

the Intermediate Value Theorem implies that f has a zero in each of the intervals $(0,1)$ and $(1,2)$; *i.e.*, f has at least two real zeros. Next, since $f''(x) = e^x$ has no real zeros, two applications of Rolle's Theorem imply that f has at most two real zeros. Therefore, f has exactly two real zeros, as required.

10. If $f(x) = x - 6\sqrt[3]{2x-3}$, then $f'(x) = 1 - 4(2x-3)^{-2/3}$, is undefined if, and only if, $x = \frac{3}{2}$, and is equal to zero if, and only if, $(2x-3)^{2/3} = 4$, *i.e.*, $2x - 3 = \pm 8$, whose solutions are $-\frac{5}{2}$ and $\frac{11}{2}$. All three critical number of f belong to the interval $[-12, 15]$. Comparing

$$f(-12) = -12 - 6(-3) = 6, \qquad f\left(-\tfrac{5}{2}\right) = -\tfrac{5}{2} - 6(-2) = \tfrac{19}{2},$$
$$f\left(\tfrac{3}{2}\right) = \tfrac{3}{2}, \qquad f\left(\tfrac{11}{2}\right) = \tfrac{11}{2} - 6(2) = -\tfrac{13}{2} \quad \text{and}$$
$$f(15) = 15 - 6(3) = -3,$$

reveals that the largest and smallest values of f on $[-12, 15]$ are, respectively, $f\left(-\frac{5}{2}\right) = \frac{19}{2}$ and $f\left(\frac{11}{2}\right) = -\frac{13}{2}$.

11. The domain of f is $\mathbb{R}\setminus\{0\}$, and $f(x) \to \pm\infty$ as $x \to 0^{\pm}$, so the y-axis is the vertical asymptote of the graph of f (and f has no global extrema). As $x \to \pm\infty$, $f(x) \to 2$, so the line defined by $y = 2$ is the horizontal asymptote of the graph of f. The axis intercepts of the graph of f are $(-\frac{1}{2}, 0)$ and $(1, 0)$. Since

$$f'(x) = \frac{3}{x^2} - \frac{3}{x^4} = \frac{3(x^2-1)}{x^4}$$

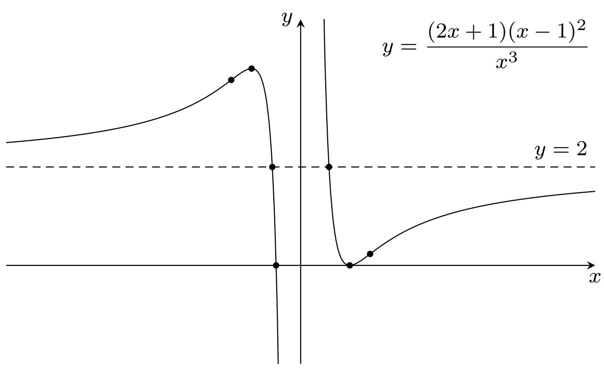
is positive if $|x| > 1$ and negative if $|x| < 1$ and $x \neq 0$, f is increasing on $(-\infty, -3)$ and on $(3, \infty)$, and decreasing on $(-3, 0)$ and on $(0, 3)$, with a local maximum at $(-1, 4)$ and a local minimum at $(1, 0)$. Next, since

$$f''(x) = 3\left\{-\frac{2}{x^3} + \frac{4}{x^5}\right\} = \frac{6(2-x^2)}{x^5}$$

is positive if $x < -\sqrt{2}$ or $0 < x < \sqrt{2}$ and negative if $-\sqrt{2} < x < 0$ or $x > \sqrt{2}$, it follows that the graph of f is concave up on $(-\infty, -\sqrt{2})$ and on $(0, \sqrt{2})$, and concave down on $(-\sqrt{2}, 0)$ and on $(\sqrt{2}, \infty)$. So the graph of f has points of inflection at $(\pm\sqrt{2}, 2 \mp \frac{5}{4}\sqrt{2})$. The graph of f meets its horizontal asymptote at $(\pm\frac{1}{3}\sqrt{3}, 2)$, which is where

$$0 = -\frac{3}{x} + \frac{1}{x^3} = \frac{1-3x^2}{x^3}.$$

Below is a sketch of the graph of f, with the horizontal asymptote drawn as a dashed line and the points of interest emphasized.



12. If x denotes the side length of the base of the square prism in the right of the figure, then the total volume of the prisms is equal to

$$V = 1^2(8-4x) + 4x^2 = (2x-1)^2 + 7,$$

where $0 \leqslant x \leqslant 2$. Since the graph of V as a function of x is an parabola which opens upward, the smallest value of V occurs where $x = \frac{1}{2}$ (at the vertex of the parabola) and the largest value of V occurs where $x = 2$ (at the endpoint farthest from the vertex of the parabola). Therefore, to minimize the volume the cut should be made 2 inches from the right side (in the figure) of the piece of paper, and to maximize the volume all of the paper should be used for the prism on the right.

13. If $f(t) = 3\sqrt{t} + 2$, then $f'(t) = 2t^{3/2} + 2t - 1$ (so that $f'(1) = 3$), and $f(t) = \frac{4}{5}t^{5/2} + t^2 - t - \frac{4}{5}$ (so that $f(1) = 0$).

14. Since

$$\begin{aligned}\frac{d}{dx}\left\{\left(2+\cos^2 x\right)\sin x\right\} &= -2\cos x \sin^2 x + \left(2+\cos^2 x\right)\cos x \\ &= \left(2(\cos^2 x - 1) + 2 + \cos^2 x\right)\cos x \\ &= 3\cos^3 x,\end{aligned}$$

it follows that

$$\int \cos^3(x)\,dx = \tfrac{1}{3}\left(2+\cos^2(x)\right)\sin(x) + C.$$

15. If $[1, 3]$ is divided into n subintervals of equal length, then the length of each subinterval is $2/n$ and the endpoints of the subintervals are $1 + \frac{2}{n}i$, for $i = 0, 1, 2, \ldots, n$. If the integrand is evaluated at the right endpoint of each subinterval, then

$$\int_1^3 \frac{\log x}{x}\,dx = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{\log\left(1+\frac{2}{n}i\right)}{1+\frac{2}{n}i},$$

since the integrand is continuous on $[1, 3]$.

16. a. Integrating term by term gives

$$\int\left\{\sqrt{\frac{6}{x}} + \sqrt{6}\,x + 6^x + e^6\right\}dx = 2\sqrt{6x} + \tfrac{1}{2}\sqrt{6}\,x^2 + \frac{6^x}{\log 6} + e^6 x + C.$$

b. Since

$$\frac{x^2+4x+3}{x^2+x} = \frac{(x+3)(x+1)}{x(x+1)} = \frac{x+3}{x} = 1 + \frac{3}{x},$$

at least if $x \neq -1$, the integral in question is equal to

$$\int_1^2 \left(1 + \frac{3}{x}\right)dx = (x + 3\log x)\,\Big|_1^2 = 1 + 3\log 2.$$

c. Since $\left(\tan x + \sec^3 x\right)\cos x = \sin x + \sec^2 x$, termwise integration gives

$$\int \left(\tan x + \sec^3 x\right)\cos(x)\,dx = -\cos x + \tan x + C.$$

d. Since $|x-2| = 2 - x$ if $x < 2$, and $|x-2| = x - 2$ if $x \geqslant 2$, it follows that

$$\begin{aligned}\int_0^5 \left(|x-2|+1\right)dx &= \int_0^2 (3-x)\,dx + \int_2^5 (x-1)\,dx \\ &= -\tfrac{1}{2}(3-x)^2\,\Big|_0^2 + \tfrac{1}{2}(x-1)^2\,\Big|_2^5 \\ &= 4 + 8 - \tfrac{1}{2} = \tfrac{23}{2}.\end{aligned}$$

17. The Fundamental Theorem of Calculus and the Chain Rule, gives

$$\frac{d}{dx}\int_2^{e^x} \frac{t}{\log(t)}\,dt = \frac{e^x}{\log(e^x)}\cdot e^x = \frac{e^{2x}}{x}.$$

18. Observe that $x^2 - x - 6 = (x+2)(x-3)$. If $p(x)$ is a polynomial,

$$\lim_{x\to\infty} \frac{p(x)}{(x-3)(x+2)} = 2, \qquad \text{and} \qquad \lim_{x\to 3} \frac{p(x)}{(x-3)(x+2)} = 1,$$

then $p(x) = (2x - \alpha)(x - 3)$, where $3 + 2 = 2\cdot 3 - \alpha$, *i.e.*, $\alpha = 1$. Therefore, $p(x) = (2x-1)(x-3)$.