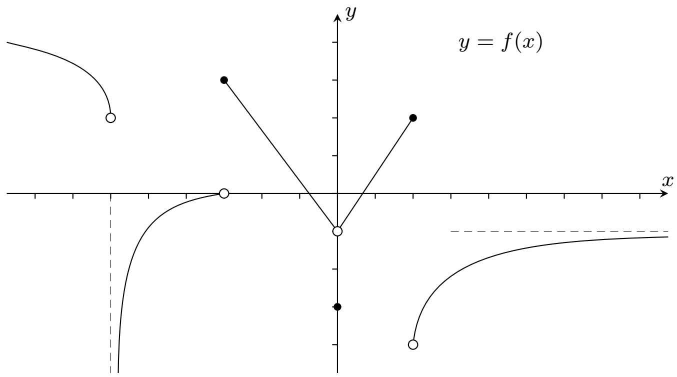
1. Here is the graph of a function f.



Evaluate the following expressions.

a. $\lim_{x\to 0} f(x)$ b. $\lim_{x\to\infty} f(x)$ c. $\lim_{x\to -3^-} f(x)$

d. $\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ e. $\lim_{x\to -6^-} f\big(f(x)\big)$

2. For each of the following, evaluate the limit or show that it is undefined.

a. $\lim_{x\to 1} \dfrac{2x^3+x-3}{x^4-1}$ b. $\lim_{t\to 2} \dfrac{2t+1-\sqrt{8t+9}}{t-2}$

c. $\lim_{x\to 4} \dfrac{x^2-4x}{x^2-8x+16}$ d. $\lim_{\vartheta\to 0} \dfrac{\sin^2(2\vartheta)}{\vartheta\tan(3\vartheta)}$ e. $\lim_{x\to\infty} \big\{2x-\sqrt{4x^2-3x}\big\}$

3. Determine all values of a and b such that the function f, defined by

$$f(x) = \begin{cases} \dfrac{\dfrac{1}{x-1}-\dfrac{1}{x}}{\dfrac{1}{x}-\dfrac{1}{x+1}} & \text{if } x \neq -1, 0, 1, \\ a & \text{if } x = -1, \\ b & \text{if } x = 0 \text{ and} \\ 5 & \text{if } x = 1, \end{cases}$$

is continuous at -1 and 0.

4. Find $\dfrac{dy}{dx}$ for each of the following.

a. $y = \dfrac{\sqrt[3]{x}-3x^4+7}{2\sqrt{x}}$ b. $y = \sin^3(x) - \log(\csc x) - \pi^{3x}$

c. $y = \dfrac{\cos(7x)-\sqrt{5x}}{\log_3(x)}$ d. $y = \dfrac{4x^6\cot^4(3x)}{\sqrt{5^x\log x}}$

e. $y^2\tan(x+y) = 4$ f. $y = (3x+2)^{xe^{2x}}$

5. Find an equation of the line tangent to the curve defined by

$$(y-5)^5 = x^2 + 2xy - 34,$$

at the point $(3, 4)$.

6. Given that

$$g(1) = 2, \qquad g'(1) = -3, \qquad g'(2) = -1, \qquad g''(2) = -4$$

and

$$f(x) = \frac{4xg(x^2)}{g'(2x)},$$

compute $f'(x)$ and $f'(1)$.

7. Find the absolute extrema of $y = x - \sqrt{x}$ over the interval $[\,0, 4\,]$.

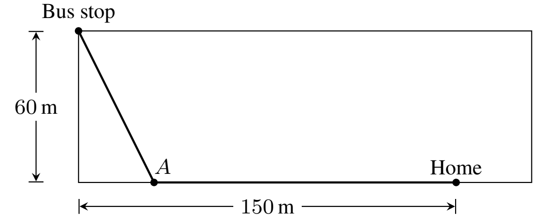
8. Stephanie is sitting on the ground 10 feet from the spot where a hot air balloon is about to land. She is watching the balloon as it travels at a steady rate of 20 feet per second towards the ground. If ϑ is the angle between the ground and her line of sight to the balloon, at what rate is this angle changing at the instant the balloon hits the ground?

9. Sketch the graph of f, where

$$f(x) = \log|x^2-9|, \quad f'(x) = \frac{2x}{x^2-9} \quad \text{and} \quad f''(x) = -2\,\frac{x^2+9}{(x^2-9)^2}.$$

Give the domain of f, and any intercepts, asymptotes, intervals of monotonicity and concavity, and all extrema and points of inflection.

10. Stephen lives next to a park. He takes a bus from work which drops him off on the opposite corner of the park (see below). Normally, he saves time by cutting diagonally through the park instead of going along the sidewalk. In the winter, however, the park is covered with snow, so he will cut across to a point A as in the figure, and continue on the sidewalk. He is able to walk $\sqrt{5}$ metres per second on the sidewalk, but only 1 metre per second through the snow. What path should he take in order to minimize his walking time?



11. Evaluate the following integrals.

a. $\displaystyle\int (3e^x + x^3 + x^e + 3)\,dx$ b. $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{1+\sin x}{\cos^2(x)}\,dx$

c. $\displaystyle\int_1^4 \frac{(x+3)(x-2)}{\sqrt{x^3}}\,dx$ d. $\displaystyle\int \big(\sin^3(x) + \sin(x)\cos^2(x)\big)\,dx$

12. The acceleration of a particle moving along a straight line is is given by $a = 2t+1$, and the initial velocity of the particle is $v_0 = -2$.
a. Find the velocity function of the particle.
b. Find the distance travelled by the particle during the time interval $[\,0, 3\,]$.
(Units are immaterial. If you wish, think of position in metres and time in seconds.)

13. Consider the definite integral

$$\int_{-1}^{5} \left(\tfrac{1}{2}x^2 - 1\right)\,dx.$$

a. Estimate the integral using a Riemann sum with 3 subintervals and midpoints as sample points.

b. Evaluate the integral as a limit of Riemann sums.

14. Given

$$F(x) = \int_x^{3x^2} \sqrt{1+\sin t}\,dt,$$

find $F'(x)$.

15. It is given that f' is continuous on $\mathbb{R}$, and that

$$f(-1) = 7, \qquad f(3) = 1 \qquad \text{and} \qquad f(9) = 9.$$

Show that the equation $\big(f'(x)\big)^2 = 1$ has at least two real solutions.

1. Inspecting the graph of f gives:
a. $\lim_{x\to 0} f(x) = -1$; b. $\lim_{x\to\infty} f(x) = -1$; c. $\lim_{x\to -3^-} f(x) = 0$;
d. the slope of the line segment joining $(0,-1)$ and $(2,2)$ is $\frac{3}{2}$, and so
$$\lim_{h\to 0}\frac{f(1+h)-f(h)}{h} = \tfrac{3}{2};$$
e. $\lim_{x\to -6^-} f\big(f(x)\big) = \lim_{t\to 2^+} f(t) = -4$.

2. a. The numerator and denominator each vanish as $x \to 1$, and factorizing by inspection gives
$$2x^3 + x - 3 = (x-1)(2x^2+2x+3)$$
and
$$x^4 - 1 = (x-1)(x^3+x^2+x+1).$$
Therefore,
$$\lim_{x\to 1}\frac{2x^2+x+3}{x^4-1} = \lim_{x\to 1}\frac{2x^2+2x+3}{x^3+x^2+x+1} = \tfrac{7}{4},$$
by independence and direct substitution.
b. If $a = 2t+1$ and $b = \sqrt{8t+9}$ then, provided $a+b \neq 0$,
$$a - b = \frac{a^2-b^2}{a+b} = \frac{4t^2-4t-8}{2t+1+\sqrt{8t+9}} = \frac{4(t+1)(t-2)}{2t+1+\sqrt{8t+9}}.$$
Therefore,
$$\lim_{t\to 2}\frac{2t+1-\sqrt{8t+9}}{t-2} = \lim_{t\to 2}\frac{4(t+1)}{2t+1+\sqrt{8t+9}} = \tfrac{6}{5},$$
by independence and direct substitution.
c. Since $x^2 - 4x = x(x-4)$ and $x^2 - 8x + 16 = (x-4)^2$, it follows that
$$\lim_{x\to 4^-}\frac{x^2-4x}{x^2-8x+16} = \lim_{\substack{x\to 4\\ x<4}}\frac{x}{x-4} = -\infty,$$
and
$$\lim_{x\to 4^+}\frac{x^2-4x}{x^2-8x+16} = \lim_{\substack{x\to 4\\ x>4}}\frac{x}{x-4} = \infty.$$
Therefore, the limit in question is undefined.
d. The definition of the tangent function gives
$$\lim_{\vartheta\to 0}\frac{\sin^2(2\vartheta)}{\vartheta\tan(3\vartheta)} = \lim_{\vartheta\to 0}\left\{\left(\frac{\sin(2\vartheta)}{2\vartheta}\right)^2\cdot\frac{3\vartheta}{\sin(3\vartheta)}\cdot\frac{4}{3\cos(3\vartheta)}\right\} = \tfrac{4}{3},$$
by arithmetical limit laws, direct substitution, and $\lim_{t\to 0}\frac{\sin t}{t} = 1$.
e. If $a = 2x$ and $b = \sqrt{4x^2-3x}$ then, provided $a+b\neq 0$,
$$a-b = \frac{a^2-b^2}{a+b} = \frac{3x}{2x+\sqrt{4x^2-3x}} = \frac{3}{2+\sqrt{4-3t}},$$
where $t = 1/x$, at least if $x > \frac{3}{4}$, *i.e.*, $0 < t < \frac{4}{3}$. Hence,
$$\lim_{x\to\infty}\left\{2x - \sqrt{4x^2-3x}\right\} = \lim_{t\to 0^+}\frac{3}{2+\sqrt{4-3t}} = \tfrac{3}{4},$$
by independence, composition and direct substitution.

3. Multiplying and dividing by $x(x-1)(x+1)$ gives
$$\frac{\dfrac{1}{x-1}-\dfrac{1}{x}}{\dfrac{1}{x}-\dfrac{1}{x+1}} = \frac{x+1}{x-1},$$
at least if $x \neq 0, \pm 1$. So f is continuous at -1 if, and only if,
$$a = f(-1) = \lim_{x\to -1} f(x) = \lim_{x\to -1}\frac{x+1}{x-1} = 0,$$
and f is continuous at 0 if, and only if,
$$b = f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{x+1}{x-1} = -1.$$

4. a. If
$$y = \frac{\sqrt[3]{x} - 3x^4 + 7}{2\sqrt{x}} = \tfrac{1}{2}x^{-1/6} - \tfrac{3}{2}x^{7/2} + \tfrac{7}{2}x^{-1/2},$$
then
$$\frac{dy}{dx} = -\tfrac{1}{12}x^{-7/6} - \tfrac{21}{4}x^{5/2} - \tfrac{7}{4}x^{-3/2}.$$
b. If $y = \sin^3(x) - \log(\csc x) - \pi^{3x} = \sin^3(x) + \log(\sin x) - e^{3x\log\pi}$, then
$$\frac{dy}{dx} = 3\sin^2(x)\cos(x) + \cot(x) - e^{3x\log\pi}\,3\log\pi.$$
c. If
$$y = \frac{\cos(7x) - \sqrt{5x}}{\log_3 x} = \log 3\cdot\frac{\cos(7x) - \sqrt{5}\,x^{1/2}}{\log x},$$
then
$$\frac{dy}{dx} = -\log 3\left\{\frac{7\sin(7x) + \frac{1}{2}\sqrt{(5/x)}}{\log x} + \frac{\cos 7x - \sqrt{5x}}{x(\log x)^2}\right\}.$$
d. If
$$y = \frac{4x^6\cot^4(3x)}{\sqrt{5^x\log x}},$$
then logarithmic differentiation gives
$$\begin{aligned}\frac{dy}{dx} &= y\frac{d}{dx}\left\{\log 4 + 6\log|x| + 4\log|\cot 3x| - \tfrac{1}{2}(\log 5)x - \tfrac{1}{2}\log|\log x|\right\}\\ &= \frac{4x^6\cot^4(3x)}{\sqrt{5^x\log x}}\left\{\frac{6}{x} - \frac{12}{\sin 3x\cos 3x} - \tfrac{1}{2}\log 5 - \frac{1}{2x\log x}\right\}.\end{aligned}$$
e. If $y^2\tan(x+y) = 4$, then differentiating implicitly with respect to x gives
$$\frac{dy}{dx} = \frac{-y^2\sec^2(x+y)}{2y\tan(x+y) + y^2\sec^2(x+y)} = \frac{-y}{y + \sin\big(2(x+y)\big)}.$$
f. If $y = (3x+2)^{xe^{2x}}$, then logarithmic differentiation gives
$$\begin{aligned}\frac{dy}{dx} &= y\frac{d}{dx}\left\{xe^{2x}\log(3x+2)\right\}\\ &= (3x+2)^{xe^{2x}-1}e^{2x}\left\{(1+2x)(3x+2)\log(3x+2) + 3x\right\}.\end{aligned}$$

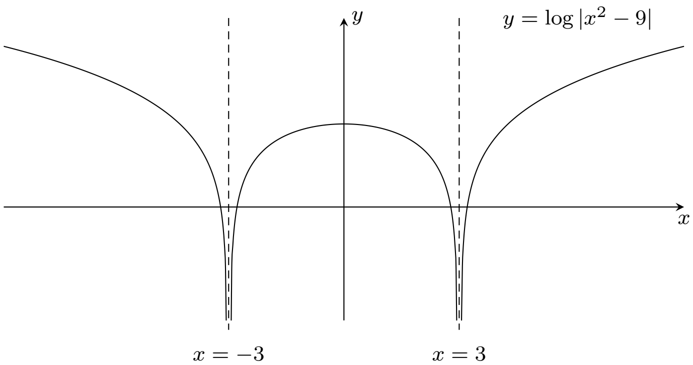
5. If $(y-5)^5 = x^2 + 2xy - 34$, *i.e.*, $(y-5)^5 - x^2 - 2xy = -34$, then implicit differentiation gives
$$\frac{dy}{dx} = \frac{2(x+y)}{5(y-5)^4 - 2x},\qquad\text{and so}\qquad \left.\frac{dy}{dx}\right|_{\substack{x=3\\ y=4}} = -14.$$
Therefore, the tangent line is defined by $14x + y = 46$, or $y = 4 - 14(x-3)$.

6. If
$$f(x) = \frac{4xg(x^2)}{g'(2x)},$$
then
$$f'(x) = 4\frac{\big(g(x^2) + 2x^2g'(x^2)\big)g'(2x) - 2xg(x^2)g''(2x)}{\big(g'(2x)\big)^2},$$
and
$$\begin{aligned}f'(1) &= 4\frac{\big(g(1) + 2g'(1)\big)g'(2) - 2g(1)g''(2)}{\big(g'(2)\big)^2}\\ &= 4\frac{(2-6)(-1) - 2(2)(-4)}{(-1)^2}\\ &= 80.\end{aligned}$$

7. If $y = x - \sqrt{x}$, then
$$\frac{dy}{dx} = 1 - \frac{1}{2\sqrt{x}},$$
which is defined on $(\,0,4\,)$ and zero if $\sqrt{x} = \frac{1}{2}$, *i.e.*, $x = \frac{1}{4}$. Since
$$f(0) = 0,\qquad f\big(\tfrac{1}{4}\big) = \tfrac{1}{4} - \tfrac{1}{2} = -\tfrac{1}{4}\qquad\text{and}\qquad f(4) = 4 - 2 = 2,$$
the absolute extrema of the given function on $[\,0,4\,]$ are $-\frac{1}{4}$ and 2.

8. If y is the height (in feet) of the balloon and ϑ is the angle (in radians) between the ground and Stephanie's line of sight to the balloon, then $\tan\vartheta = \frac{1}{10}y$, and so
$$\sec^2\vartheta\,\frac{d\vartheta}{dt} = \frac{1}{10}\,\frac{dy}{dt} = \tfrac{1}{10}(-20) = -2.$$
Since $\sec 0 = 1$, it follows that the angle between Stephanie's line of sight and the ground is decreasing at a rate of 2 radians per second as the balloon hits the ground.

9. The domain of f is $\mathbb{R} \setminus \{\pm 3\}$ and the vertical asymptotes of the graph are defined by $x = \pm 3$. The graph has no horizontal or oblique asymptotes. Since $f(0) = \log 9 = 2\log 3$, the y-intercept of the graph of f is $(0, 2\log 3)$. The zeros of f occur where $|x^2 - 9| = 1$, *i.e.*, where $x = \pm 2\sqrt{2}, \pm\sqrt{10}$. Since $f(-x) = f(x)$, f it is sufficient to consider positive values of x in the remaining analysis. As $f'(x) < 0$ if $0 < x < 3$ and $f'(x) > 0$ if $x > 3$, f is decreasing on $(\,0, 3\,)$ and increasing on $(\,3, \infty\,)$, and therefore f has no local extrema on $(\,0, \infty\,)$ (by symmetry the y-intercept is a local, but not a global, maximum). Since $f''(x) < 0$ if $x > 0$ and $x \neq 3$, the graph of f is concave down on $(\,0, 3\,)$ and on $(\,3, \infty\,)$, with no points of inflection. In the sketch of the curve (which is symmetric in the y-axis), the vertical asymptotes are drawn as dashed lines.



10. If x denotes the distance (measured in units of 30 metres) between A and the bottom left corner of the park in the figure, and t denotes Stephen's total walking time (measured in half minutes), then

$$t = \sqrt{x^2 + 4} + \frac{5 - x}{\sqrt{5}},$$

where $0 \leqslant x \leqslant 5$, and so

$$\frac{dt}{dx} = \frac{x}{\sqrt{x^2 + 4}} - \frac{1}{\sqrt{5}},$$

whose positive zero occurs where $5x^2 = x^2 + 4$, *i.e.*, $4x^2 = 4$, or $x = 1$. Since $\frac{dt}{dx} < 0$ if $x < 1$ and $\frac{dt}{dx} > 0$ if $x > 1$, the First Derivative Test for global extreme values implies that the minimum value of t occurs where $x = 1$. This means that to minimize his walking time Stephen should walk to a point 30 metres from the end of the park near, but opposite to, the bus stop, in order to minimize his waling time. (Observe that the same result could be deduced from Snellius' principle.)

11. a. Integrating by inspection gives

$$\int (3e^x + x^3 + x^e + 3)\,dx = 3e^x + \tfrac{1}{4}x^4 + \tfrac{1}{e+1}x^{e+1} + 3x + C.$$

b. Since

$$\frac{1 + \sin x}{\cos^2(x)} = \frac{1}{\cos^2(x)} + \frac{\sin x}{\cos^2(x)} = \sec^2(x) + \sec(x)\tan(x),$$

it follows that

$$\int_0^{\frac{1}{4}\pi} (\sec^2 x + \sec x \tan x)\,dx = (\tan x + \sec x)\Big|_0^{\frac{1}{4}\pi} = \sqrt{2}.$$

c. Since

$$\frac{(x+3)(x-2)}{\sqrt{x^3}}\,dx = \frac{x^2 + x - 6}{x^{3/2}} = (x^{1/2} + x^{-1/2} - 6x^{-3/2}),$$

it follows that

$$\begin{aligned} \int_1^4 \frac{(x+3)(x-2)}{\sqrt{x^3}}\,dx &= \left(\tfrac{2}{3}x^{3/2} + 2x^{1/2} + 12x^{-1/2}\right)\Big|_1^4 \\ &= \tfrac{14}{3} + 2 - 6 \\ &= \tfrac{2}{3}. \end{aligned}$$

d. Since $\sin^3(x) + \sin(x)\cos^2(x) = (\sin^2(x) + \cos^2(x))\sin(x) = \sin(x)$, the integral in question is equal to

$$\int \sin(x)\,dx = -\cos(x) + C.$$

12. a. If the acceleration of the particle is $a = v' = 2t + 1$ and the initial velocity of the particle is $v(0) = -2$, then the velocity function of the particle is

$$v = t^2 + t - 2 = (t+2)(t-1).$$

b. Since $v < 0$ if $0 < t < 1$, and $v > 0$ if $t > 1$, the distance travelled by the particle during the first three seconds is equal to

$$\begin{aligned} \int_0^3 |v|\,dt &= \int_1^0 (t^2 + t - 2)\,dt + \int_1^3 (t^2 + t - 2)\,dt \\ &= \left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 - 2t\right)\Big|_1^0 + \left(\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 - 2t\right)\Big|_1^3 \\ &= 2 \cdot \tfrac{7}{6} + \tfrac{15}{2} \\ &= \tfrac{59}{6}. \end{aligned}$$

13. a. If $[\,-1, 5\,]$ is divided into three subintervals of equal length, then the width of each subinterval is 2, the midpoints of the subintervals are 0, 2, 4, and the values of the integrand at the midpoints are -1, 1, 7. The corresponding midpoint sum is equal to $2(-1 + 1 + 7) = 14$.

b. If $[\,-1, 5\,]$ is divided into n subintervals of equal length, then the width of each subinterval is $6/n$, the endpoints of the subintervals are $-1 + 6j/n$, and the values of the integrand at these endpoints are

$$\frac{1}{2}\Big(-1 + \frac{6}{n}j\Big)^2 - 1 = -\frac{1}{2} - \frac{6}{n}j + \frac{18}{n^2}j^2,$$

for $j = 0, 1, 2, \ldots, n$. The corresponding right endpoint sum is

$$\begin{aligned} \mathscr{R}_n = \frac{6}{n}\sum_{j=1}^{n}\Big(-\frac{1}{2} - \frac{6}{n}j + \frac{18}{n^2}j^2\Big) &= 6\Big\{-\frac{1}{2} - \frac{6}{n^2}\sum_{j=1}^{n} j + \frac{18}{n^3}\sum_{j=1}^{n} j^2\Big\} \\ &= 6\Big\{-\frac{1}{2} - \frac{6}{n^2}\cdot\frac{1}{2}n(n+1) + \frac{18}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)\Big\} \\ &= 6\Big\{-\frac{1}{2} - 3\Big(1 + \frac{1}{n}\Big) + 6\Big(1 + \frac{1}{n}\Big)\Big(1 + \frac{1}{2n}\Big)\Big\}. \end{aligned}$$

Therefore,

$$\int_{-1}^{5} \left(\tfrac{1}{2}x^2 - 1\right)dx = \lim_{n\to\infty} \mathscr{R}_n = 6\left\{-\tfrac{1}{2} - 3 + 6\right\} = 15.$$

14. The interval additivity of the definite integral, the (first) Fundamental Theorem of Calculus (and the Chain Rule) imply that

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_0^{3x^2} \sqrt{1 + \sin t}\,dt - \frac{d}{dx}\int_0^x \sqrt{1 + \sin t}\,dt \\ &= 6x\sqrt{1 + \sin(3x^2)} - \sqrt{1 + \sin x}. \end{aligned}$$

15. Since f' is continuous on $\mathbb{R}$, the Mean Value Theorem may be applied to f, and the Intermediate Value Theorem to f', on any closed interval of positive length. Applying the Mean Value Theorem to f on the interval $[\,-1, 3\,]$ yields a real number ξ such that $-1 < \xi < 3$, and

$$-6 = f(3) - f(-1) = 4f'(\xi), \qquad \text{or} \qquad f'(\xi) = -\tfrac{3}{2}.$$

Applying the Mean Value Theorem to f on the interval $[\,3, 9\,]$ yields a real number η such that $3 < \eta < 9$, and

$$8 = f(9) - f(3) = 6f'(\eta), \qquad \text{or} \qquad f'(\eta) = \tfrac{4}{3}.$$

As -1 and 1 are between $-\frac{3}{2}$ and $\frac{4}{3}$, the Intermediate Value Theorem, applied to f' on the interval $[\,\xi, \eta\,]$ yields real numbers λ and μ such that $\xi < \lambda < \eta$, $\xi < \mu < \eta$, $f'(\lambda) = -1$ and $f'(\mu) = 1$. Therefore, λ and μ are two (since they must be different) solutions of the equation $\big(f'(x)\big)^2 = 1$.