1. Evaluate the following limits.

a. $\displaystyle\lim_{x\to -2}\frac{x^2+2x}{x^4+6x-4}$ b. $\displaystyle\lim_{x\to -2^-}\frac{x+1}{4-x^2}$ c. $\displaystyle\lim_{x\to -\infty}\frac{\sqrt{2x^2+1}}{3x-5}$

d. $\displaystyle\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{2-\sqrt{x}}$ e. $\displaystyle\lim_{x\to 0}\frac{\tan x-\sin(2x)}{x}$

2. Find a and b so that f is continuous on $\mathbb{R}$, where

$$f(x)=\begin{cases}\dfrac{x+1}{x^2+x} & \text{if } x<-1,\\ ax+b & \text{if } -1\leqslant x<2 \quad\text{and}\\ x^2-2 & \text{if } 2\leqslant x.\end{cases}$$

3. Sketch the graph of a function f which satisfies all conditions below.

- $f(-5)=0$, $f(-\frac{1}{2})=0$ and $f(3)$ is undefined.
- $\lim_{x\to -4}f(x)=\infty$, $\lim_{x\to 1^-}f(x)=\infty$ and $\lim_{x\to 1^+}f(x)=-\infty$.
- $\lim_{x\to\infty}f(x)=-3$.

4. Use the definition of the derivative to find $\frac{dy}{dx}$, where $y=\sqrt{x^2+1}$.

5. Evaluate $\displaystyle\lim_{h\to 0}\frac{\sin\left(\frac{1}{2}\pi+h\right)-1}{h}$.

6. Find (the x and y coordinates of) each point on the parabola defined by $y=2x^2+1$, at which the tangent line passes through the point $(1,-5)$.

7. Find $\frac{dy}{dx}$ for each of the following.

a. $y=\cos^2(x)\sec(x^2)+\log_3(x)+\pi^{e^{\pi}}$ b. $y=\dfrac{\tan^2(e^x-3)}{\log(3x^2+5)}$

c. $y=\left(\log(\cos(e^{3x+7}))\right)^6$ d. $y=(\cot x)^{\sin x}$ e. $y=\sqrt[4]{\dfrac{x^5\sin^2(x)}{(x-5)^6}}$

8. Find an equation of the line tangent to the curve defined by

$$x^2+2xy+4y^2=13$$

at the point $(-1,2)$.

9. Prove that the equation $x^3+33x=8$ has exactly one real root.

10. For the function f, defined by

$$f(x)=\frac{2}{x^2}-\frac{9}{x^4},$$

give

a. equations of all asymptotes of the graph of f,

b. the intervals of monotonicity of f, and

c. all extreme values of f.

11. Sketch the graph of $y=x(x-5)^{2/3}$, given that

$$\frac{dy}{dx}=\frac{5(x-3)}{3(x-5)^{1/3}}\qquad\text{and}\qquad\frac{d^2y}{dx^2}=\frac{10(x-6)}{9(x-5)^{4/3}}.$$

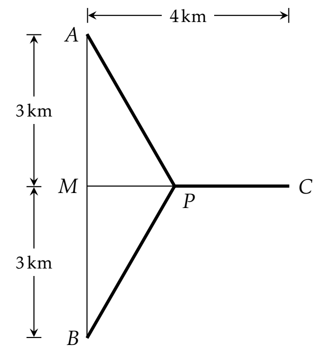
Make sure that your solution includes the domain, and all intercepts, asymptotes, intervals of monotonicity and concavity, local extreme values and points of inflection.

12. Find the absolute maximum and minimum values of

$$f(t)=4t^3-5t^2-8t+3$$

on the closed interval $[-1,1]$.

13. Factory A is six kilometres north of factory B, while power plant C is four kilometres east of the point M midway between A and B. Power is to be delivered to these two factories through a cable that will run from C to P, where it will split into two branches, one going to A and the other going to B. (See the figure below.) How far from M should P be located in order to minimise the total length of the cable?



14. Find the position function of a particle moving in a straight line with acceleration $a=6t+4$, initial velocity $v_0=-6$ and initial position $s_0=9$.

15. Compute the definite integral

$$\int_0^2(2x^3-1)\,dx$$

as a limit of Riemann sums.

16. Evaluate each of the following integrals.

a. $\displaystyle\int(e^x+x^3+3^x+e^3)\,dx$ b. $\displaystyle\int\frac{(2x+\sqrt{x})^2}{x^3}\,dx$

c. $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\sec\vartheta\tan\vartheta\csc\vartheta\,d\vartheta$ d. $\displaystyle\int_{-3}^{2}|2x-1|\,dx$

17. Evaluate the limit

$$\lim_{n\to\infty}\left\{\frac{1}{n}\left(\sqrt[3]{\frac{1}{n}}+\sqrt[3]{\frac{2}{n}}+\sqrt[3]{\frac{3}{n}}\cdots+\sqrt[3]{\frac{n}{n}}\right)\right\}$$

by expressing it as a definite integral.

18. Use the (first) fundamental theorem of calculus to find the second derivative with respect to x of

$$\int_{\log x}^{x}te^t\,dt.$$

19. Mark each statement as true or false. Justify your answers.

a. The graph of $y=\dfrac{x^3-4x}{x-2}$ has a vertical asymptote where $x=2$.

b. If f is continuous at a then f must be differentiable at a.

c. If $\int f(x)\,dx=x^2\log x+C$, then $f(x)=x+2x\log x$.

d. $\displaystyle\int_{\pi}^{\pi}\sqrt{\tan x}\,dx=0$.

e. If $\lim_{x\to 0^+}f(x)=0$ then either $\displaystyle\lim_{x\to 0^+}\frac{1}{f(x)}=\infty$ or else $\displaystyle\lim_{x\to 0^+}\frac{1}{f(x)}=-\infty$.

1. a. The numerator and denominator vanish as $x \to -2$, and factorising by inspection gives

$$x^2 + 2x = x(x+2), \qquad \text{and} \qquad x^4 + 6x - 4 = (x+2)(x^3 - 2x^2 + 4x - 2).$$

Therefore, the limit in question is equal to

$$\lim_{x\to -2} \frac{x}{x^3 - 2x^2 + 4x - 2} = \frac{-2}{-8-8-8-2} = \tfrac{1}{13}.$$

b. If $x \to -2$ and $x < -2$, then $x+1 \to -1$, $4-x^2 \to 0$ and $4-x^2 < 0$; therefore,

$$\lim_{x\to -2^-} \frac{x+1}{4-x^2} = \infty.$$

c. Extracting dominant powers of x gives (remember that $\sqrt{x^2} = -x$ if $x < 0$)

$$\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to -\infty} \frac{-\sqrt{2+x^{-2}}}{3-5x^{-1}} = -\tfrac{1}{3}\sqrt{2}.$$

d. If $x > 0$ then

$$\frac{1}{x} - \frac{1}{4} = \frac{4-x}{4x} = \frac{(2-\sqrt{x})(2+\sqrt{x})}{4x}.$$

Therefore,

$$\lim_{x\to 4} \frac{\frac{1}{x} - \frac{1}{4}}{2-\sqrt{x}} = \lim_{x\to 4} \frac{2+\sqrt{x}}{4x} = \frac{2+2}{4\cdot 4} = \tfrac{1}{4}.$$

e. Since

$$\tan x - \sin(2x) = \sin x \sec x - 2\sin x \cos x = (\sin x)(\sec x - 2\cos x),$$

it follows that

$$\lim_{x\to 0} \frac{\tan x - \sin(2x)}{x} \lim_{x\to 0}\left\{ \frac{\sin x}{x} \cdot (\sec x - 2\cos x) \right\} = 1(1-2) = -1.$$

2. As $f(x) = 1/x$ if $x < -1$, and on the each of the intervals $[-1,2)$ and $[2,\infty)$ f is defined by a polynomial, for f to be continuous on $\mathbb{R}$ it is necessary and sufficient that f be continuous at -1 and at 2. Now

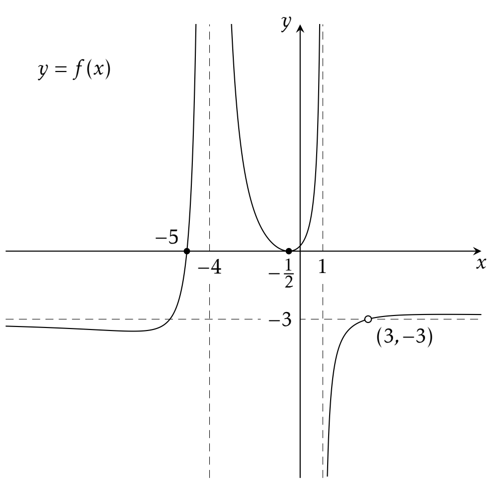
$$\lim_{x\to -1^-} f(x) = -1, \qquad \lim_{x\to -1^+} f(x) = f(-1) = -a+b,$$
$$\lim_{x\to 2^-} f(x) = 2a+b \qquad \text{and} \qquad \lim_{x\to 2^+} f(x) = f(2) = 2.$$

So f is continuous at -1 and at 2 if, and only if, $a-b=1$ and $2a+b=2$, which gives (adding the equations) $3a=3$, or $a=1$, and hence $b=0$ (using either equation). Therefore, f is continuous at -1 and at 2, or equivalently, f is continuous on on $\mathbb{R}$, if, and only if, $a=1$ and $b=0$.

3. The function f defined by

$$f(x) = \frac{3(x+5)(2x+1)^2(x-3)}{4(x+4)^2(1-x)(x-3)}$$

is an example of such a function, whose graph is displayed below.



4. If $y = \sqrt{x^2+1}$, then $y^2 = x^2+1$, so the factorisation of a difference of squares gives

$$y' - y = \frac{y'^2 - y^2}{y'+y} = \frac{x'^2 - x^2}{y'+y} = \frac{(x'-x)(x'+x)}{y'+y}.$$

Since $y' \to y$ as $x' \to x$, it follows that

$$\frac{dy}{dx} = \lim_{x'\to x} \frac{y'-y}{x'-x} = \lim_{x'\to x} \frac{x'+x}{y'+y} = \frac{2x}{2y} = \frac{x}{\sqrt{x^2+1}}.$$

5. From the definition of the derivative, it follows that

$$\lim_{h\to 0} \frac{\sin\left(\frac{1}{2}\pi + h\right) - 1}{h} = \frac{d}{dx}\left\{\sin x\right\}\bigg|_{x=\frac{1}{2}\pi} = \cos x\bigg|_{x=\frac{1}{2}\pi} = \cos\left(\tfrac{1}{2}\pi\right) = 0.$$

6. A tangent to the curve $y = 2x^2+1$ at (x,y) contains $(1,-5)$ if, and only if,

$$\frac{y+5}{x-1} = \frac{dy}{dx}, \qquad \text{or} \qquad \frac{2x^2+6}{x-1} = 4x,$$

which is equivalent to $x^2+3 = 2x^2-2x$, or $0 = x^2-2x-3 = (x+1)(x-3)$. Therefore, the tangent line to the given parabola at the points $(-1,3)$ and $(3,19)$—and no other points—pass through $(1,-5)$.

7. a. If $y = \cos^2(x)\sec(x^2) + \log_3(x) + \pi^{e^\pi} = \dfrac{\cos^2(x)}{\cos(x^2)} + \dfrac{\log x}{\log 3} + \pi^{e^\pi}$, then

$$\frac{dy}{dx} = \frac{-2\cos(x)\sin(x)}{\cos(x^2)} + \frac{2x\cos^2(x)\sin(x^2)}{\cos^2(x^2)} + \frac{1}{x\log 3}.$$

b. If $y = \dfrac{\tan^2(e^x-3)}{\log(3x^2+5)}$, then

$$\frac{dy}{dx} = \frac{2e^x\tan(e^x-3)\sec^2(e^x-3)}{\log(3x^2+5)} - \frac{6x\tan^2(e^x-3)}{(3x^2+5)\left(\log(3x^2+5)\right)^2}.$$

c. If $y = \left(\log(\cos(e^{3x+7}))\right)^6$, then

$$\frac{dy}{dx} = -18e^{3x+7}\left(\log(\cos(e^{3x+7}))\right)^5\tan(e^{3x+7}).$$

d. If $y = (\cot x)^{\sin x}$, then logarithmic differentiation gives

$$\begin{aligned}\frac{dy}{dx} &= y\frac{d}{dx}\left\{\log y\right\} = y\frac{d}{dx}\left\{(\sin x)\log(\cot x)\right\}\\ &= (\cot x)^{\sin x}\left\{(\cos x)\log(\cot x) - \sec x\right\}.\end{aligned}$$

e. If $y = \sqrt[4]{\dfrac{x^5\sin^2(x)}{(x-5)^6}}$, then logarithmic differentiation gives

$$\begin{aligned}\frac{dy}{dx} &= y\frac{d}{dx}\left\{\log|y|\right\} = \tfrac{1}{4}y\frac{d}{dx}\left\{5\log|x| + 2\log|\sin(x)| - 6\log|x-5|\right\}\\ &= \tfrac{1}{4}\sqrt[4]{\frac{x^5\sin^2(x)}{(x-5)^6}}\left\{\frac{5}{x} + 2\cot(x) - \frac{6}{x-5}\right\}.\end{aligned}$$

8. If $x^2 + 2xy + 4y^2 = 13$, then implicit differentiation gives

$$\frac{dy}{dx}\bigg|_{\substack{x=-1\\ y=\ 2}} = -\frac{x+y}{x+4y}\bigg|_{\substack{x=-1\\ y=\ 2}} = -\frac{-1+2}{-1+8} = -\tfrac{1}{7},$$

so the tangent line to the curve at the point $(-1,2)$ is defined by $x + 7y = 13$.

9. Let $f(x) = x^3 + 33x - 8$; f is differentiable on $\mathbb{R}$, so the Intermediate Value Theorem and the Mean Value Theorem apply to f on any closed interval of positive length. Since

$$f(0) = -8 < 0 \qquad \text{and} \qquad f(1) = 26 > 0,$$

the Intermediate Value Theorem implies that there is a real number ξ such that $0 < \xi < 1$ and $f(\xi) = 0$. If $\xi' \neq \xi$, then the Mean Value Theorem implies that there is a real number η between ξ and ξ' such that

$$f(\xi') - f(\xi) = f'(\eta)(\xi' - \xi), \qquad \text{or} \qquad f(\xi') = (3\eta^2 + 33)(\xi' - \xi),$$

and hence $|f(\xi')| \geqslant 33|\xi' - \xi| > 0$. Therefore, ξ is the unique real zero of f. This shows that the equation in question has exactly one real solution.

10. If

$$f(x) = \frac{2}{x^2} - \frac{9}{x^4} = \frac{2x^2 - 9}{x^4}, \quad \text{then} \quad f'(x) = -\frac{4}{x^3} + \frac{36}{x^5} = \frac{4(9 - x^2)}{x^5}.$$

a. The domain of f is $\mathbb{R} \setminus \{0\}$, $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to\pm\infty} f(x) = 0$, so the asymptotes of the graph of f are defined by $x = 0$ and $y = 0$.

b. Since $f'(x) > 0$ if $0 < x < 3$ or $x < -3$ and $f'(x) < 0$ if $-3 < x < 0$ or $3 < x$, f is increasing on the intervals $(-\infty, -3]$ and $(0, 3]$, and decreasing on the intervals $[-3, 0)$ and $[3, \infty)$.

c. From Part b it follows that $f(\pm 3) = \frac{1}{9}$ is the (local and global) maximum value of f, and that f has no (local or global) minimum values.

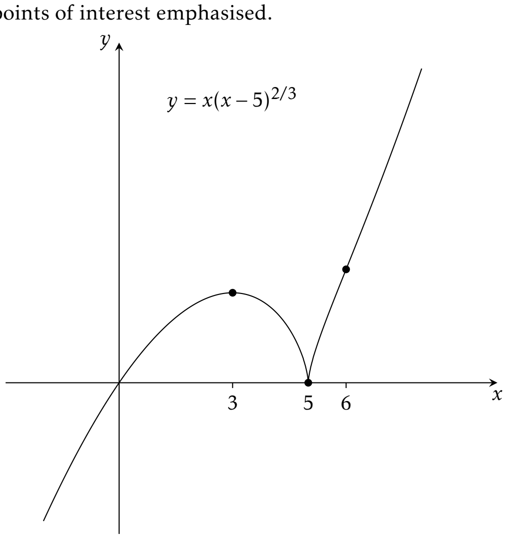
11. Since $y = x(x-5)^{2/3}$ is a continuous function of x on $\mathbb{R}$, the curve has no vertical asymptotes. Also, since $y = x^{5/3}(1 - 5x^{-1})^{2/3}$ if $x \neq 0$, the curve has no horizontal or oblique asymptotes, and no global extreme values. The axis intercepts of the curve are $(0,0)$ and $(5,0)$. The derivative,

$$\frac{dy}{dx} = \frac{5(x-3)}{3(x-5)^{1/3}},$$

is positive if $x < 3$ or $5 < x$, negative if $3 < x < 5$, zero if $x = 3$ and undefined if $x = 5$. So y is increasing on $(-\infty, 3]$ and on $[5, \infty)$, decreasing on $[3, 5]$, and has a local maximum at $(3, 3\sqrt[3]{4})$ and a local minimum at $(5, 0)$. The second derivative,

$$\frac{d^2y}{dx^2} = \frac{10(x-6)}{9(x-5)^{4/3}},$$

is positive if $6 < x$, negative if $x < 5$ or $5 < x < 6$, zero if $x = 6$ and undefined if $x = 5$. So the curve is concave up on the interval $[6, \infty)$, concave down on the interval $(-\infty, 5]$ and $[5, \infty)$, and has a point of inflection at $(6, 6)$. Below is a sketch of the curve (not to scale—the x-axis is dilated by a factor of 2), with the points of interest emphasised.



12. If $f(t) = 4t^3 - 5t^2 - 8t + 3$, then

$$f'(t) = 12t^2 - 10t - 8 = 2(3t - 4)(2t + 1),$$

so the critical numbers of f are $-\frac{1}{2}$ and $\frac{4}{3}$, of which only the first lies in the closed interval $[-1, 1]$. Since

$$f(-1) = -4 - 5 + 8 + 3 = 2, \qquad f\left(-\tfrac{1}{2}\right) = -\tfrac{1}{2} - \tfrac{5}{4} + 4 + 3 = \tfrac{21}{4}$$

and

$$f(1) = 4 - 5 - 8 + 3 = -6,$$

the largest and smallest values of f on $[-1, 1]$ are, respectively, $\frac{21}{4}$ and -6.

13. If x is the distance between M and P, and y is the distance between P and C (each measured in kilometres), then $0 \leqslant x \leqslant 4$ and $x + y = 4$, so $\frac{dy}{dx} = -1$. The total length of the cable is (by Pythagoras' formula)

$$\ell = 2\sqrt{x^2 + 3^2} + y, \qquad \text{and hence} \qquad \frac{d\ell}{dx} = \frac{2x}{\sqrt{x^2 + 3^2}} - 1,$$

which is equal zero if, and only if, $4x^2 = x^2 + 3^2$, or $x = \sqrt{3}$ (because $x \geqslant 0$). Since $\frac{d\ell}{dx} < 0$ if $0 \leqslant x < \sqrt{3}$ and $\frac{d\ell}{dx} > 0$ if $\sqrt{3} < x$, the First Derivative Test for global extrema implies that the minimum length of the cable occurs if P is $\sqrt{3}$ kilometres east of M.

14. If $a = \frac{dv}{dt} = 6t + 4$ and $v_0 = -6$, then $v = 3t^2 + 4t - 6$ (by inspection). Likewise, since $v = \frac{ds}{dt}$, if the initial position of the particle is $s_0 = 9$ then $s = t^3 + 2t^2 - 6t + 9$.

15. If $[0, 2]$ is divided into k closed subintervals of equal length, then the length of each subinterval is $\frac{2}{k}$, the endpoints of the subintervals are $\frac{2}{k}j$, and the values of the integrand at these endpoint are $2\left(\frac{2}{k}j\right)^3 - 1 = \frac{16}{k^3}j^3 - 1$, for $j = 0, 1, 2, \ldots, k$. The corresponding right endpoint Riemann sum is

$$\mathscr{R}_k = \frac{2}{k}\sum_{j=1}^{k}\left\{\frac{16}{k^3}j^3 - 1\right\} = \frac{2}{k}\left\{\frac{16}{k^3}\sum_{j=1}^{k} j^3 - k\right\} = 2\left\{\frac{16}{k^4} \cdot \frac{1}{4}k^2(k+1)^2 - 1\right\}$$
$$= 2\left\{4\left(1 + \frac{1}{k}\right)^2 - 1\right\}.$$

Therefore,

$$\int_0^2 (2x^3 - 1)\,dx = \lim_{k\to\infty} \mathscr{R}_k = 2(4 - 1) = 6.$$

16. a. Integrating by inspection (and noting that $3^x = e^{(\log 3)x}$) gives

$$\int (e^x + x^3 + 3^x + e^3)\,dx = e^x + \tfrac{1}{4}x^4 + 3^x(\log 3)^{-1} + e^3 x + C.$$

b. Since $(2x + \sqrt{x})^2 = 4x^2 + 4x\sqrt{x} + x$, it follows that

$$\frac{(2x + \sqrt{x})^2}{x^3} = 4x^{-1} + 4x^{-3/2} + x^{-2}.$$

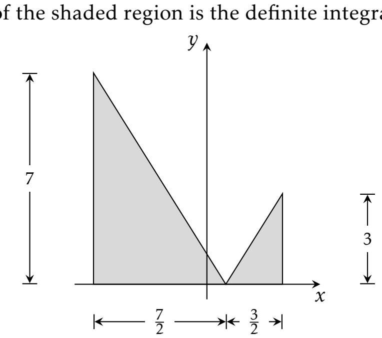
Therefore,

$$\int \frac{(2x + \sqrt{x})^2}{x^3}\,dx = 4\log x - 8x^{-1/2} - x^{-1} + C.$$

c. Since $\sec\vartheta \tan\vartheta \csc\vartheta = \sec^2\vartheta$, it follows that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sec\vartheta \tan\vartheta \csc\vartheta \,d\vartheta = \tan\vartheta \Big|_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} = \sqrt{3} - \tfrac{1}{3}\sqrt{3} = \tfrac{2}{3}\sqrt{3}.$$

d. Below is a sketch of the graph of $y = |2x - 1|$ on $[-3, 2]$ (not to scale), in which the area of the shaded region is the definite integral in question.



By interpreting the integral in terms of area, it follows that

$$\int_{-3}^{2} |2x-1|\,dx = \tfrac{49}{4} + \tfrac{9}{4} = \tfrac{29}{2}.$$

17. By inspection,

$$\lim_{n\to\infty}\left\{\frac{1}{n}\sum_{\nu=1}^{n}\sqrt[3]{\frac{\nu}{n}}\right\} = \int_0^1 \sqrt[3]{x}\,dx = \tfrac{3}{4}x^{4/3}\Big|_0^1 = \tfrac{3}{4}.$$

18. The interval additivity of the definite integral, the (first) fundamental theorem of calculus and the chain rule imply that

$$\frac{d}{dx}\int_{\log x}^{x} te^t\,dt = \frac{d}{dx}\int_0^x te^t\,dt - \frac{d}{dx}\int_0^{\log x} te^t\,dt = xe^x - (\log x)x\cdot x^{-1} = xe^x - \log x.$$

Therefore,

$$\frac{d^2}{dx^2}\int_{\log x}^{x} te^t\,dt = \frac{d}{dx}\left\{xe^x - \log x\right\} = e^x(x-1) - x^{-1}.$$

19. a. If $x \neq 2$, then

$$y = \frac{x^3-4x}{x-2} = \frac{x(x-2)(x+2)}{x-2} = x(x+2), \qquad \text{and hence} \qquad \lim_{x\to 2} y = 8.$$

So the curve has a hole, not a vertical asymptote, where $x = 2$. Therefore, the statement in question is false.

b. A function which is continuous at a need not be differentiable at a. For example, if $f(x) = |x-a|$, then f is continuous on $\mathbb{R}$ but not differentiable at a. Therefore, the statement in question is false.

c. Since

$$\frac{d}{dx}\left\{x^2\log x\right\} = 2x\log x + x^2\cdot x^{-1} = 2x\log x + x,$$

it follows that

$$\int (x + 2x\log x)\,dx = x^2\log x + C.$$

Therefore, the statement in question is true.

d. Since $\sqrt{\tan x}$ is defined if $x = \pi$, it follows that

$$\int_{\pi}^{\pi} \sqrt{\tan x}\,dx = 0.$$

Therefore, the statement in question is true.

e. If $f(x) = x\cos(\pi/x)$, then $-x \leqslant f(x) \leqslant x$ for $x > 0$, and hence $\lim_{x\to 0^+} f(x) = 0$. However, if n is a positive integer, then

$$\frac{1}{f(1/n)} = n\cos(\pi n) = n, \qquad \text{if } n \text{ is even},$$

so $\lim_{x\to 0^+} f(x)$ is not $-\infty$, and likewise

$$\frac{1}{f(1/n)} = -n, \qquad \text{if } n \text{ is odd},$$

so $\lim_{x\to 0^+} f(x)$ is not ∞. Therefore, the statement in question is false.