1. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 3}\frac{\frac{3}{7}-\frac{x}{3x-2}}{x-3}$ b. $\displaystyle\lim_{x\to -3^+}\frac{x^2-9}{x^2+6x+9}$ c. $\displaystyle\lim_{x\to 0}\frac{x+\sin(x)}{\tan(x)}$

d. $\displaystyle\lim_{x\to 0}\frac{2\cos^2(x)-7\cos(x)+5}{\cos(x)-1}$ e. $\displaystyle\lim_{x\to -\infty}\frac{\sqrt{x^2+1}-(7-2x)}{x-2}$ f. $\displaystyle\lim_{x\to\infty}\frac{2x+\sin(x)}{3x}$

2. Consider the function

$$f(x)=\begin{cases}\dfrac{x+3}{x^2+7x+12} & \text{if } x<-3,\\ x^2+3 & \text{if } -3\leqslant x<-2 \text{ and}\\ \dfrac{x+9}{x+3} & \text{if } x>-2.\end{cases}$$

a. Identify all points of discontinuity, and state whether the discontinuity is removable, jump or infinite.

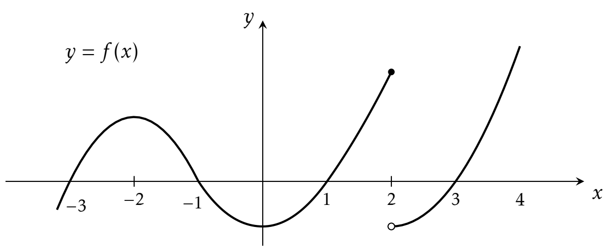
b. Find all horizontal asymptotes of the graph of f.

3. Consider the function $f(x)=\sqrt{x^2-5}$.

a. Find $f'(x)$ using the limit definition of the derivative.

b. Find an equation of the tangent line to the graph of f where $x=3$.

4. Given the following graph of a function, draw a rough sketch of a graph of its derivative. (**Note**: This question is, in essence, asking you to BS.)



5. Find $\dfrac{dy}{dx}$ for each of the following. You need not simplify your answers.

a. $y=1+2x^3+\dfrac{4}{\sqrt[5]{x}}+6^x+\tan\left(\frac{7}{8}\pi\right)+9\log_{10}(x)$ b. $y=\sec(3x^2+2)\cos(8xe^x)$

c. $y=\log\left(\sqrt[5]{\dfrac{(2x^3+1)\sin(x)}{(4x-1)^6}}\right)$ d. $\sqrt{4x^2+3y^3}=x+y$ e. $y=\frac{5x}{1+x^{\sin(x)}}$

6. Show that if f, g and h are differentiable, and h is not zero, then

$$\left(\frac{fg}{h}\right)'=\frac{f'gh+fg'h-fgh'}{h^2}.$$

7. Find the 81^{st} derivative of $f(x)=\cos(10x)$.

8. Find all points on the graph of $y=e^x\cos(x)$ at which the tangent line is horizontal.

9. Barbara is flying a kite in a large field. The kite is 50 feet above the ground and moves horizontally away from Barbara at a speed of 4 feet per second. At what rate is the angle of elevation of the string decreasing when 100 feet of string have been let out?

10. Show that the equation $\cos(3x)+2016x=0$ has exactly one solution.

11. Find the absolute maximum and minimum values of $f(x)=(x^2+2x)^{2/3}$ on the closed interval $[-3,2]$.

12. Given

$$f(x)=\frac{(x-1)^2}{(x+1)^2},\qquad f'(x)=\frac{4(x-1)}{(x+1)^3}\qquad\text{and}\qquad f''(x)=\frac{8(2-x)}{(x+1)^4},$$

find the domain of f, all intervals of monotonicity and concavity, and all extreme values and points of inflection.

13. Sketch the graph of a function f such that:

- f is continuous at all real numbers besides -1;
- $f(-2)=f(1)=f(3)=0$, $f(0)=-1$, $f(2)=2$ and $f(4)=2$;
- the graph of f has a vertical asymptote defined by $x=-1$;
- $f(x)\to -3$ as $x\to-\infty$ and $f(x)\to 4$ as $x\to\infty$;
- $f'(x)>0$ if $x<-1$, or $-1<x<2$ or $x>3$, and $f'(x)<0$ if $2<x<3$;
- $f''(x)>0$ if $x<-1$ or $1<x<2$ or $2<x<4$, and $f''(x)<0$ if $-1<x<1$ or $x>4$.

Indicate clearly all asymptotes, extreme values and points of inflection.

14. Find all points on the curve defined by $y=x^2$ which are closest to $(0,3)$.

15. Find $f(x)$, if $f''(x)=x^2+\sin(x)-2e^x+3$, $f'(0)=1$ and $f(0)=4$.

16. Evaluate each of the following integrals.

a. $\displaystyle\int\left(\frac{\sqrt{x^5}}{x^3}-e^x-\cos(3)\right)dx$ b. $\displaystyle\int_1^e\frac{x^2+2x+1}{x^3+x^2}\,dx$

c. $\displaystyle\int\sec^2(x)(1+\sin(x))\,dx$ d. $\displaystyle\int_0^4|2x-3|\,dx$

17. Evaluate the limit

$$\lim_{n\to\infty}\frac{1^5+2^5+3^5+\cdots+n^5}{n^6}$$

by expressing it as a definite integral.

18. Given

$$g(x)=\int_2^{\sqrt{x}}\frac{t}{\log(1+t)}\,dt,$$

find $g(4)$ and $g'(x)$.

19. Given that $\displaystyle\int_1^3 f=8$, $\displaystyle\int_2^5 f=-3$ and $\displaystyle\int_1^2 f=\int_2^3 f$, find $\displaystyle\int_2^3 f$ and $\displaystyle\int_1^5 f$.

20. If $\displaystyle\lim_{x\to 4}f(x)=0$ and $\displaystyle\lim_{x\to 4}\frac{f(x)}{g(x)-\pi}=10$, then what is $\displaystyle\lim_{x\to 4}g(x)$?

1. a. Since

$$\frac{3}{7} - \frac{x}{3x-2} = \frac{2(x-3)}{7(3x-2)},$$

the limit is equal to

$$\lim_{x\to 3} \frac{2}{7(3x-2)} = \tfrac{2}{49}.$$

b. If $x \to -3^+$, then $x-3 \to -6$, $x+3 \to 0^+$, so

$$\lim_{x\to -3^+} \frac{x^2-9}{x^2+6x+9} = \lim_{x\to -3^+} \frac{x-3}{x+3} = -\infty.$$

c. Since $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, the limit is equal to

$$\lim_{x\to 0} \left\{ \cos(x) \left(\frac{x}{\sin(x)} + 1 \right) \right\} = 2.$$

d. Let $c = \cos(x)$; since $2c^2 - 7c + 5 = (c-1)(2c-5)$, the limit is equal to

$$\lim_{c\to 1} \frac{(2c-5)(c-1)}{c-1} = -3.$$

e. Extracting dominant powers gives

$$\lim_{x\to -\infty} \frac{2 - 7/x - \sqrt{1+1/x^2}}{1-2/x} = 1,$$

for $\sqrt{x^2} = -x$ if $x < 0$.

f. If $\varepsilon > 0$ and $x > \dfrac{1}{3\varepsilon}$, then $\left|\dfrac{\sin x}{3x}\right| \leqslant \dfrac{1}{3x} < \varepsilon$, so the limit is equal to

$$\lim_{x\to\infty} \left(\frac{2}{3} + \frac{\sin x}{3x} \right) = \tfrac{2}{3}.$$

2. The definition of f should be corrected so that

$$f(x) = \frac{x+3}{x^2+7x+12} = \frac{1}{x+4} \qquad \text{if} \quad x < -3 \quad \textbf{and} \quad x \neq -4,$$

and one acceptable response is to refuse the question on the grounds that the given definition of f is incoherent. Alternatively, note that the domain of f is $\mathbb{R} \setminus \{-4, -2\}$, contrary to what is explicitly declared in the question, that $f(x) \to \pm\infty$ as $x \to -4^{\pm}$, and that

$$\lim_{x\to -3^-} f(x) = 1, \qquad \lim_{x\to -3^+} f(x) = f(-3) = 12$$

and

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = 7.$$

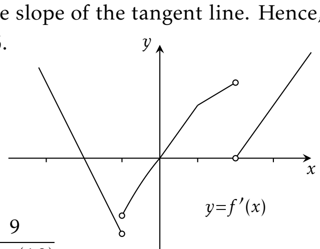
So f is continuous on the intervals $(-\infty, -4)$, $(-4, -3)$, $[-3, -2)$ and $(-2, \infty)$. The function f has a infinite discontinuity at -4, a jump discontinuity at -3 and a removable discontinuity at -2. As $x \to -\infty$, $f(x) \to 0^-$, and as $x \to \infty$, $f(x) \to 1^+$, so the horizontal asymptotes of the graph of f are defined by $y = 0$ and $y = 1$.

3. If $y = \sqrt{x^2-5}$ and $y' = \sqrt{x'^2-5}$, then $y'^2 - y^2 = x'^2 - x^2$, or equivalently $(y'-y)(y'+y) = (x'-x)(x'+x)$. Therefore,

$$\frac{dy}{dx} = \lim_{x'\to x} \frac{y'-y}{x'-x} = \lim_{x'\to x} \frac{x'+x}{y'+y} = \frac{2x}{2y} = \frac{x}{y} = \frac{x}{\sqrt{x^2-5}}.$$

If $x = 3$ then $y = 2$, and $\frac{dy}{dx} = \frac{x}{y} = \frac{3}{2}$ is the slope of the tangent line. Hence, the tangent line is defined by $3x - 2y = 5$.

4. A sketch of the graph of the actual derivative is shown to the right, with unit lengths marked along the x-axis.



5. a. $\dfrac{dy}{dx} = 6x^2 - \frac{4}{5}x^{-6/5} + 6^x \log(6) + \dfrac{9}{x\log(10)}$

b. $\dfrac{dy}{dx} = \dfrac{6x\sin(3x^2+2)\cos(8xe^x)}{\cos^2(3x^2+2)} - \dfrac{8e^x(x+1)\sin(8xe^x)}{\cos(3x^2+2)}$

c. $\dfrac{dy}{dx} = \dfrac{1}{5}\left(\dfrac{6x^2}{2x^3+1} + \dfrac{\cos(x)}{\sin(x)} - \dfrac{24}{4x-1} \right)$

d. $\dfrac{dy}{dx} = -\dfrac{8x - 2(x+y)}{9y^2 - 2(x+y)} = \dfrac{2(3x-y)}{2(x+y) - 9y^2}$

e. $\dfrac{dy}{dx} = \dfrac{5}{1+x^{\sin(x)}} - \dfrac{5x^{\sin(x)}(\sin(x) + x\cos(x)\log(x))}{\left(1+x^{\sin(x)}\right)^2}.$

6. The product and quotient rules give

$$\left(\frac{fg}{h}\right)' = \frac{f'g + fg'}{h} - \frac{fgh'}{h^2} = \frac{f'gh + fg'h - fgh'}{h^2}.$$

7. Since $f^{(4)}(x) = 10^4\cos(10x)$, it follows (after twenty iterations) that

$$f^{(81)}(x) = \frac{d}{dx}\left\{10^{80}\cos(10x)\right\} = -10^{81}\sin(10x).$$

8. If $y = e^x\cos(x)$, then

$$\frac{dy}{dx} = e^x(\cos(x) - \sin(x)),$$

which is zero where $\cos(x) = \sin(x)$, whose solutions are $\frac{1}{4}(4k+1)\pi$, where k is an integer, and the corresponding values of y fall into two families. If $x = \frac{1}{4}(8k+1)\pi$ then $y = \frac{1}{2}\sqrt{2}\exp\left(\frac{1}{4}(8k+1)\pi\right)$, and if $x = \frac{1}{4}(8k+5)\pi$ then $y = -\frac{1}{2}\sqrt{2}\exp\left(\frac{1}{4}(8k+5)\pi\right)$.

9. If Barbara is x feet from the point on the ground directly beneath the kite and ϑ is the angle of elevation of the string, then

$$x = 50\cot(\vartheta), \qquad \text{so} \qquad \frac{dx}{d\vartheta} = \frac{-50}{\sin^2(\vartheta)}, \qquad \text{or} \qquad \frac{d\vartheta}{dx} = -\frac{1}{50}\sin^2(\vartheta).$$

When $x = 100$, $\sin\vartheta = \frac{50}{100} = \frac{1}{2}$, and thus

$$\frac{d\vartheta}{dt} = \frac{d\vartheta}{dx}\frac{dx}{dt} = -\frac{1}{50}\left(\frac{1}{2}\right)^2 \cdot 4 = -\frac{1}{50}, \qquad \text{since} \qquad \frac{dx}{dt} = 4.$$

Therefore, the angle of elevation of the string is decreasing by $\frac{1}{50}$ radians per second when one hundred feet of string have been released.

10. If $f(x) = \cos(3x) + 2016x$, then f is differentiable (and therefore also continuous) everywhere, so the intermediate value theorem and the mean value theorem apply to f on any closed interval of positive length. Since $f(-1) = \cos(-3) - 2016 < 0$ and $f(0) = \cos(0) = 1 > 0$, the intermediate value theorem implies that $f(r) = 0$ for some r such that $-1 < r < 0$. If $s \neq r$ then the mean value theorem implies that there is a real number t between s and r such that $|f(s)| = |f'(t)(s-r)| = |(-3\sin(t) + 2016)(s-r)| \geqslant 2013|s-r| > 0$. Therefore, the equation $f(x) = 0$ has exactly one solution (namely, r).

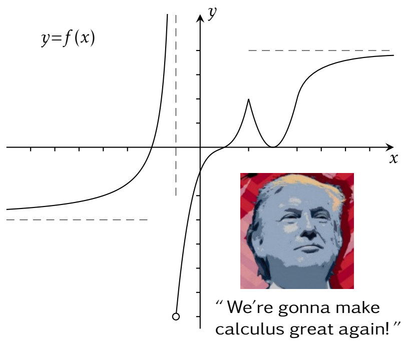
11. As $f(x) = (x^2+2x)^{2/3} \geqslant 0$ for all real values of x, it is apparent that $f(0) = f(-2) = 0$ is the least value of f on $[-3, 2]$. Now,

$$f'(x) = \tfrac{2}{3}(x^2+2x)^{-1/3}(2x+2) = \tfrac{4}{3}(x+1)(x^2+2x)^{-1/3},$$

so it remains to compare the values $f(-3) = \sqrt[3]{9}$, $f(-1) = 1$ and $f(2) = 4$, of which 4 is the largest.

12. The domain of f is $\mathbb{R} \setminus \{-1\}$. Since $f'(x) > 0$ if $x < -1$ or $x > 1$, and $f'(x) < 0$ if $-1 < x < 1$, f is increasing on the intervals $(-\infty, -1)$ and $[1, \infty)$, and decreasing on the interval $(-1, 1]$, with a local (and global) minimum value at its x-intercept $(1, 0)$. Since $f''(x) > 0$ if $x < -1$ or $-1 < x < 2$, and $f''(x) < 0$ if $x > 2$, the graph of f is concave up on the intervals $(-\infty, -1)$ and $(-1, 2]$, and concave down on the interval $[2, \infty)$, with a point of inflection at $\left(2, \frac{1}{9}\right)$.

13. Here is a sketch of part of the graph of one such function, with unit lengths marked along the coordinate axes.



14. The distance between $(0,3)$ and (x,x^2) is

$$\sqrt{x^2+(x^2-3)^2} = \tfrac{1}{2}\sqrt{4x^4-20x^2+36} = \tfrac{1}{2}\sqrt{(2x^2-5)^2+11},$$

whose smallest value occurs where $2x^2=5$, *i.e.*, at the points $\left(\pm\tfrac{1}{2}\sqrt{10}, \tfrac{5}{2}\right)$. (The minimum distance is $\tfrac{1}{2}\sqrt{11}$.)

15. If $f''(x) = x^2+\sin(x)-2e^x+3$ and $f'(0)=1$, then

$$f'(x) = \tfrac{1}{3}x^3 - \cos(x) - 2e^x + 3x + 4.$$

Further, if $f(0)=4$ then

$$f(x) = \tfrac{1}{12}x^4 - \sin(x) - 2e^x + \tfrac{3}{2}x^2 + 4x + 6.$$

16. a. Revising the integrand gives

$$\int (x^{-1/2} - e^x - \cos 3)\,dx = 2\sqrt{x} - e^x - (\cos 3)x + \alpha.$$

b. Revising the integrand gives

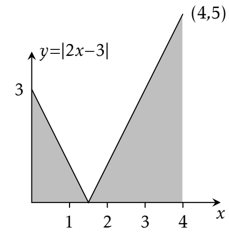
$$\int_1^e \frac{x+1}{x^2}\,dx = \int_1^e (x^{-1}+x^{-2})\,dx = (\log(x)-x^{-1})\Big|_1^e = 2-e^{-1}.$$

c. Revising the integrand gives

$$\int (\sec^2(x)+\sec(x)\tan(x))\,dx = \tan x + \sec x + \gamma.$$

d. The integral is the sum of the areas of two triangles, one of base and average height $\frac{3}{2}$, and the other of base and average height $\frac{5}{2}$, as in the figure below. Therefore,

$$\int_0^4 |2x-3|\,dx = \left(\tfrac{3}{2}\right)^2 + \left(\tfrac{5}{2}\right)^2 = \tfrac{17}{2}.$$



17. Extracting the factor n from the denominator gives

$$\frac{1^5+2^5+3^5+\cdots+n^5}{n^6} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^5.$$

This is a Riemann sum of x^5 for a partition of the interval $[0,1]$ into n subintervals of equal length, taking the right endpoints of the intervals as sample points. Therefore,

$$\lim_{n\to\infty}\frac{1^5+2^5+3^5+\cdots+n^5}{n^6} = \int_0^1 x^5\,dx = \tfrac{1}{6}.$$

18. Plainly, $g(4)=0$ (it is the definite integral of a continuous function on an interval of length zero). Next, the first fundamental theorem of calculus and the chain rule give

$$g'(x) = \frac{\sqrt{x}}{\log(1+\sqrt{x})}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\log(1+\sqrt{x})}.$$

19. By interval additivity,

$$\int_1^2 f = \int_2^3 f = \tfrac{1}{2}\int_1^3 f = 4, \qquad \text{and} \qquad \int_1^5 f = \int_1^2 f + \int_2^5 f = 4-3 = 1.$$

20. Since

$$\lim_{x\to 4} g(x) - \pi = \tfrac{1}{10}\lim_{x\to 4}\left\{\left(g(x)-\pi\right)\frac{f(x)}{g(x)-\pi}\right\} = \lim_{x\to 4} f(x) = 0,$$

it follows that $\lim_{x\to 4} g(x) = \pi$.