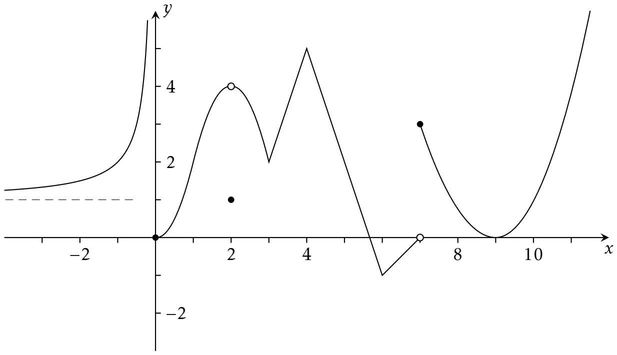
1. Below is the sketch of the graph of a function f. (**Note**: This question is essentially asking you to BS.)



Determine each of the following.

a. $\lim\limits_{x\to-\infty} f(x)$ b. $\lim\limits_{x\to0^-}\big(5-f(x)\big)$ c. $\lim\limits_{x\to7^-}\dfrac{1}{f(x)}$ d. $\lim\limits_{h\to0}\dfrac{f(5+h)-f(5)}{h}$

e. All values of x at which f is not differentiable.

2. Evaluate each of the following limits.

a. $\lim\limits_{x\to5}\dfrac{2\sqrt{x-1}-\sqrt{x^2-9}}{2x^2-9x-5}$ b. $\lim\limits_{x\to3}\dfrac{9-x^2}{x^2-|6-5x|}$

c. $\lim\limits_{x\to0}\dfrac{\tan(\pi x)}{x}$ d. $\lim\limits_{x\to0^+}\sqrt{\sqrt{x}+x^2}\cos\left(\dfrac{\pi}{x}\right)$

e. $\lim\limits_{x\to0}\dfrac{x-\cos(x)}{x^2}$

3. List all asymptotes of the curve defined by $y=\dfrac{1+e^x}{e^x-5}$.

4. Find all values of a, if any, so that the function f is continuous on $\mathbb{R}$.

$$f(x)=\begin{cases} ax^2-5 & \text{if } x<2,\\ a^2 & \text{if } x=2 \quad \text{and}\\ x^2+ax+7 & \text{if } x>2.\end{cases}$$

5. State the definition of the derivative and use it to compute $f'(x)$, where

$$f(x)=\frac{1}{2x^2+5}.$$

6. Find $\frac{dy}{dx}$ for each of the following. Do not simplify your answers.

a. $y=\sqrt[3]{x^5}-\dfrac{3^x}{\log(5x)}+6e^{7\pi}-\sec\left(\dfrac{8}{x}\right)$ b. $\sin(x^2y)=ye^x$

c. $y=\dfrac{\tan^2(x)}{(8x^2-7)\sqrt{5x+1}}$ d. $y=\big(e^x(x^2-5)+\log(x)\big)^9$

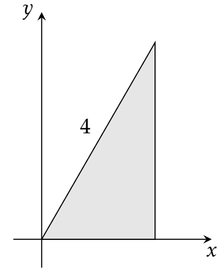
e. $y=\left(\dfrac{\csc(x)}{3}\right)^{\log_2(x)}$

7. Find an expression for $f^{(57)}(x)$, given that $f(x)=xe^x+10x^{32}$.

8. Find all points on the graph of $y=\log(x^3-3x^2-9x+10)$, at which the tangent line is horizontal.

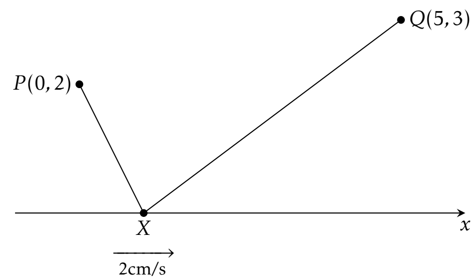
9. Sketch the graph of $y=x^4+4x^3$. Be sure that your solution includes all intercepts, intervals of monotonicity and concavity, and all extreme values and points of inflection.

10. A line segment of length 4 rotates counter-clockwise about the origin from the positive x axis to the positive y axis, and a triangle is formed by connecting the tip of the line segment to the x axis perpendicularly (as in the figure below). What is the maximum area of the triangle?



11. Find the extreme values of $f(x)=(2x)^{2/3}(10-x)+1$ on $\left[-4,\tfrac{1}{2}\right]$.

12. In the figure below, X is moving along the x-axis towards the right at a rate of 2cm/s. Find the rate at which $|PX|+|QX|$, the sum of the distances between X and the points $P(0,2)$ and $Q(5,3)$, is changing as X passes the point $(3,0)$.



13. Use the mean value theorem to show that $\sqrt[3]{1+x}<1+\frac{1}{3}x$ if $x>0$.

14. The position of an object is given by $x=\sin^2(t)-\sin(t)$, for $t\geqslant 0$.

a. Express the velocity of the object as a function of time t.

b. Find the distance travelled by the object while $\pi\leqslant t<2\pi$.

15. Evaluate each of the following integrals.

a. $\displaystyle\int\left(\frac{2}{3x^2}-\frac{4}{x}+6^x-\pi^2\right)dx$ b. $\displaystyle\int\frac{(2-\sqrt{x})^2}{2\sqrt{x}}\,dx$

c. $\displaystyle\int_{\frac{1}{3}\pi}^{\frac{1}{4}\pi}\frac{\tan(x)}{\sec(x)}\,dx$ d. $\displaystyle\int_{-4}^{4}\left(|x|-\sqrt{16-x^2}\right)dx$

16. Given that $f''(x)=\dfrac{4}{x^{2/3}}-\dfrac{3}{x^{1/2}}$, $f'(1)=12$ and $f(1)=15$, find $f(x)$.

17. Compute $\displaystyle\int_0^2(4x-3x^2)\,dx$ as a limit of Riemann sums.

18. Compute and simplify $f'(x)$, where $f(x)=\displaystyle\int_1^{1/x}\frac{dt}{e^{1/t}+1}$.

1. Nothing precise can be said about a function from a picture of its graph as given, although certain guesses are at least plausible.
a. It is plausible that the curve approaches the line defined by $y = 1$ as $x \to -\infty$, and that the limit is 1.
b. One might guess that $f(x) \to \infty$ as $x \to 0^-$, so that the limit is $-\infty$.
c. One might presume that $f(x) \to 0^-$ as $x \to 7^-$, so that the limit is $-\infty$.
d. The graph of f might be a straight line where $4 \leqslant x \leqslant 6$, and the slope might be roughly -3 (the ordinate appears to drop by 6, as the abscissa appears to increases by 2). So the limit could be -3.
e. The graph appears to be discontinuous where $x = 0$, 2 and 7, where f would not be differentiable. It may be that f is not differentiable at 3, 4 and 6, although it is easy to give functions whose graphs look just like those parts of f and are differentiable at those numbers.

2. a. If $a = 2\sqrt{x-1}$ and $b = \sqrt{x^2-9}$, then $a^2 - b^2 = 5 + 4x - x^2 = (5-x)(1+x)$. As, $2x^2 - 9x - 5 = (x-5)(2x+1)$, the factorization of a difference of squares gives

$$\lim_{x\to 5} \frac{a-b}{2x^2-9x-5} = \lim_{x\to 5} \frac{-(x+1)}{(a+b)(2x+1)} = \frac{-6}{(4+4)(11)} = -\tfrac{3}{44}.$$

b. If $x > \frac{6}{5}$ then $x^2 - |6-5x| = 6 - 5x + x^2 = (2-x)(3-x)$. Therefore, since $9 - x^2 = (3-x)(3+x)$, it follows that

$$\lim_{x\to 3} \frac{9-x^2}{x^2-|6-5x|} = \lim_{x\to 3} \frac{3+x}{2-x} = -6.$$

c. Since $\sin(\vartheta)/\vartheta \to 1$ and $\cos(\vartheta) \to 1$ as $\vartheta \to 0$, it follows that

$$\lim_{x\to 0} \frac{\tan(\pi x)}{x} = \pi \lim_{x\to 0} \left\{ \frac{\sin(\pi x)}{\pi x} \cdot \frac{1}{\cos(\pi x)} \right\} = \pi.$$

d. If $0 < x < 1$ then $-\sqrt{2\sqrt{x}} \leqslant \sqrt{x^2+\sqrt{x}}\cos(\pi/x) \leqslant \sqrt{2\sqrt{x}}$, so plainly

$$\lim_{x\to 0^+} \sqrt{x^2+\sqrt{x}}\cos\left(\frac{\pi}{x}\right) = 0.$$

(The absolute value of the expression is $< \varepsilon$ if $0 < x < 1$ and $0 < 4x < \varepsilon^4$.)
e. Since $x - \cos(x) \to -1$ and $x^2 \to 0^+$ as $x \to 0$, it follows that

$$\lim_{x\to 0} \frac{x - \cos(x)}{x^2} = -\infty.$$

3. As $x \to \log(5)^{\pm}$, $1 + e^x \to 6$ and $e^x - 5 \to 0^{\pm}$, so $y \to \pm\infty$. Also,

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{e^{-x}+1}{1-5e^{-x}} = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{1+e^x}{e^x-5} = -\tfrac{1}{5}.$$

So the asymptotes of the curve are defined by $x = \log(5)$, $y = 1$ and $y = -\frac{1}{5}$.

4. The equation

$$f(2) = \lim_{x\to 2^-} f(x) \qquad \text{is equivalent to} \qquad a^2 = 4a - 5,$$

or $(a-2)^2 = -1$. So there is no real value of a for which f is continuous from the left at 2, let alone continuous everywhere.

5. The derivative of y with respect to x is defined by

$$\frac{dy}{dx} = \lim_{x'\to x} \frac{y'-y}{x'-x},$$

with its conventional domain (the set of all values of x for which the limit is defined).

If $z = 2x^2 + 5$ and $y = f(x) = 1/z$, then $z' - z = 2(x'^2 - x^2) = 2(x'-x)(x'+x)$ and $y' - y = (z - z')/(z'z) = -2(x'-x)(x'+x)/(z'z)$. Therefore,

$$f'(x) = \frac{dy}{dx} = \lim_{x'\to x} \frac{y'-y}{x'-x} = -\lim_{x'\to x} \frac{2(x'+x)}{z'z} = -\frac{4x}{z^2} = -\frac{4x}{(2x^2+5)^2}.$$

6. a. If $y = x^{5/3} - 3^x/\log(5x) + 6e^{7\pi} - 1/\cos(8/x)$, then

$$\frac{dy}{dx} = \tfrac{5}{3}x^{2/3} - \frac{3^x\log(3)}{\log(5x)} + \frac{3^x}{x(\log(5x))^2} + \frac{8\sin(8/x)}{x^2\cos^2(8/x)}.$$

b. If $\sin(x^2y) - ye^x = 0$, then implicit differentiation gives

$$\frac{dy}{dx} = -\frac{2xy\cos(x^2y) - ye^x}{x^2\cos(x^2y) - e^x}.$$

c. If $y = \tan^2(x)(8x^2-7)^{-1}(5x+1)^{-1/2}$ then logarithmic differentiation gives

$$\frac{dy}{dx} = y\frac{d}{dx}\left\{\log|y|\right\} = \frac{\tan^2(x)}{(8x^2-7)\sqrt{5x+1}} \left\{ \frac{2}{\sin(x)\cos(x)} - \frac{16x}{8x^2-7} - \frac{5}{2(5x+1)} \right\}.$$

d. If $y = \left(e^x(x^2-5) + \log(x)\right)^9$, then

$$\frac{dy}{dx} = 9\left(e^x(x^2-5) + \log(x)\right)^8 \left(e^x(x^2+2x-5) + 1/x\right).$$

e. If $y = (3\sin x)^{-\log_2(x)} = (3\sin x)^{-\frac{\log(x)}{\log(2)}}$, then logarithmic differentiation gives

$$\frac{dy}{dx} = y\frac{d}{dx}\left\{\log(y)\right\} = -\frac{(3\sin x)^{-\log_2(x)}}{\log(2)} \cdot \left\{ \frac{\log(3\sin x)}{x} + \frac{\log(x)\cos(x)}{\sin(x)} \right\}.$$

7. Since $\frac{d}{dx}\left\{(x+a)e^x\right\} = (x+a+1)e^x$, if $a \in \mathbb{R}$ and $\frac{d^n}{dx^n}(x^{32}) = 0$ if $n > 32$, it follows that

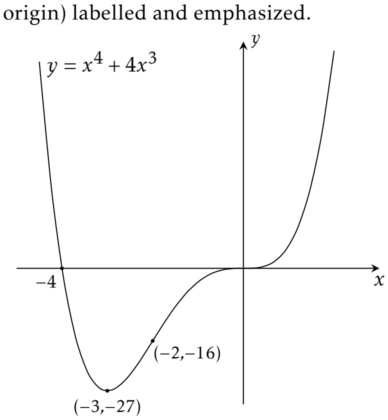
$$\frac{d^{57}}{dx^{57}}\left(xe^x + 10x^{32}\right) = e^x(x+57).$$

8. If $y = \log(x^3 - 3x^2 - 9x + 10)$, then

$$\frac{dy}{dx} = \frac{3x^2-6x-9}{x^3-3x^2-9x+10}, \qquad \text{provided} \qquad x^3 - 3x^2 - 9x + 10 > 0.$$

The numerator of the derivative is $3(x+1)(x-3)$. Since 3 does not belong to the domain of the function, the tangent line to the curve horizontal at the point $(-1, \log(15))$, and nowhere else.

9. If $f(x) = x^4 + 4x^3 = x^3(x+4)$, then $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$ and $f''(x) = 12x^2 + 24x = 12x(x+2)$. The derivative $f'(x)$ is negative if $x < -3$ and is positive or zero if $x > -3$, so f is decreasing on $(-\infty, -3\,]$, increasing on $[\,-3, \infty)$ and has a local (and global) minimum value at the point $(-3, -27)$. The second derivative $f''(x)$ is positive if $x < -2$ or else $x > 0$ and is negative if $-2 < x < 0$, so the graph of f is concave up on $(-\infty, -2\,]$ and on $[\,0, \infty)$, is concave down on $[\,-2, 0\,]$, and has points of inflection at the origin and at the point $(-2, -16)$. The axis intercepts of the curve are the origin and $(-4, 0)$, and $f(x) \to \infty$ as $x \to \pm\infty$. Below is a sketch of the curve (not to scale, the y axis is contracted by a factor of 10), with the points of interest (apart from the origin) labelled and emphasized.



10. If ϑ denotes the angle between the segment and the positive x axis, then the area of the triangle is $\frac{1}{2} \cdot 4\cos(\vartheta) \cdot 4\sin(\vartheta) = 4\sin(2\vartheta)$, for $0 < \vartheta < \frac{1}{2}\pi$, and its largest value is 4, which occurs where $\vartheta = \frac{1}{4}\pi$. (*i.e.*, where $\sin(2\vartheta) = 1$).

11. If $f(x) = (2x)^{2/3}(10-x)+1$ then

$$f'(x) = \tfrac{4}{3}(2x)^{-1/3}(10-x) - (2x)^{2/3} = \frac{4(10-x)-3(2x)}{3(2x)^{1/3}} = \frac{10(4-x)}{3(2x)^{1/3}},$$

so zero is the only critical number of f in $\left(-4, \frac{1}{2}\right)$. Since $f(-4) = 57$, $f(0) = 1$ and $f\left(\frac{1}{2}\right) = \frac{21}{2}$, the extreme values of f on $\left[-4, \frac{1}{2}\right]$ are 1 and 57.

12. The sum of the distances is $s = |PX| + |QX| = \sqrt{x^2+4} + \sqrt{(x-5)^2+9}$ by Pythagoras' formula, where X is $(x,0)$, so

$$\frac{ds}{dt} = \frac{2x}{\sqrt{x^2+4}} + \frac{2(x-5)}{\sqrt{(x-5)^2+9}}, \qquad \text{since} \qquad \frac{dx}{dt} = 2.$$

If x is 3 then $|PX| = |QX| = \sqrt{13}$, and thus $\frac{ds}{dt} = \left(2\cdot 3 + 2(-2)\right)\sqrt{13}^{-1} = \frac{2}{13}\sqrt{13}$. Therefore, the sum $|PX| + |QX|$ is increasing at a rate of $\frac{2}{13}\sqrt{13}$ cm/s as X passes the point $(3,0)$.

13. If $f(x) = (1+x)^{1/3}$ then f is differentiable (and thus continuous) on $(-1,\infty)$. If $x > 0$, the mean value theorem applied to f on $[0,x]$ yields a real number ξ such that $0 < \xi < x$ and $f(x) = f(0) + f'(\xi)(x-0)$, or equivalently, $(1+x)^{1/3} = 1 + \frac{1}{3}x(1+\xi)^{-2/3}$. Since $\xi > 0$, it follows that $(1+\xi)^{-2/3} < 1$, and therefore $(1+x)^{1/3} < 1 + \frac{1}{3}x$, as required.

14. a. If the position of the object is $x = \sin^2(t) - \sin(t)$, then the velocity of the object is

$$\frac{dx}{dt} = 2\sin(t)\cos(t) - \cos(t) = (2\sin(t)-1)\cos(t).$$

b. Since $\frac{dx}{dt} > 0$ if $\pi < t < \frac{3}{2}\pi$, the object travels $x\left(\frac{3}{2}\pi\right) - x(\pi) = 2$ units while $\pi \leqslant t \leqslant \frac{3}{2}\pi$. Since $\frac{dx}{dt} < 0$ if $\frac{3}{2}\pi < t < 2\pi$, the object travels $x\left(\frac{3}{2}\pi\right) - x(2\pi) = 2$ units while $\frac{3}{2}\pi \leqslant t \leqslant 2\pi$. So the object travels 4 units while $\pi \leqslant t \leqslant 2\pi$.

15. a. Integrating by inspection gives

$$\int \left(\frac{2}{3x^2} - \frac{4}{x} + 6^x - \pi^2\right) dx = -\frac{2}{3x} - 4\log|x| + \frac{6^x}{\log(6)} - \pi^2 x + \alpha.$$

b. Since $(2-\sqrt{x})^2 = 4 - 4\sqrt{x} + x$, it follows that

$$\int \frac{(2-\sqrt{x})^2}{2\sqrt{x}}\,dx = \int \left(\frac{2}{\sqrt{x}} - 2 + \tfrac{1}{2}\sqrt{x}\right) dx = 4\sqrt{x} - 2x + \tfrac{1}{3}x\sqrt{x} + \beta.$$

c. Since $\tan(x)/\sec(x) = \sin(x)$, the integral is equal to

$$\int_{\frac{1}{3}\pi}^{\frac{1}{4}\pi} \sin(x)\,dx = -\cos(x)\Big|_{\frac{1}{3}\pi}^{\frac{1}{4}\pi} = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{2}.$$

d. The definite integral of $|x|$ on $[-4,4]$ is the area of a square of side 4, or 16, and the definite integral of $\sqrt{16-x^2}$ on $[-4,4]$ is the area of a semicircle of radius 4, or 8π. Therefore,

$$\int_{-4}^{4} \left(|x| - \sqrt{16-x^2}\right) dx = 16 - 8\pi.$$

16. If $f''(x) = 4x^{-2/3} - 3x^{-1/2}$ and $f'(1) = 12$ then, by inspection and the mean value theorem, $f'(x) = 12x^{1/3} - 6\sqrt{x} + 6$. Likewise, if $f(1) = 15$ then $f(x) = 9x^{4/3} - 4x^{3/2} + 6x + 4$, at least if $x > 0$ (also if $x = 0$, provided f is continuous from the right at 0).

17. If the interval $[0,2]$ is divided into k subintervals of equal length, then the length of each subinterval is $2/k$, and the endpoints of the subintervals are $2j/k$, for $j = 0,1,2,\dots,k$. The Riemann sum obtained by evaluating the integrand at the right endpoint of each subinterval is

$$\begin{aligned}\mathscr{R}_k &= \frac{2}{k}\sum_{j=1}^{k}\left\{4\cdot\frac{2j}{k} - 3\left(\frac{2j}{k}\right)^2\right\} = \frac{16}{k^2}\sum_{j=1}^{k} j - \frac{24}{k^3}\sum_{j=1}^{k} j^2 \\ &= \frac{16}{k^2}\cdot\frac{1}{2}k(k+1) - \frac{24}{k^3}\cdot\frac{1}{6}k(k+1)(2k+1) \\ &= 8\left(1+\frac{1}{k}\right) - 8\left(1+\frac{1}{k}\right)\left(1+\frac{1}{2k}\right).\end{aligned}$$

Therefore,

$$\int_0^2 (4x - 3x^2)\,dx = \lim_{k\to\infty} \mathscr{R}_k = 8 - 8 = 0.$$

18. By the first fundamental theorem of calculus and the chain rule, if

$$f(x) = \int_1^{1/x} \frac{dt}{1+e^{1/t}} \qquad \text{then} \qquad f'(x) = \frac{1}{1+e^x}\cdot\frac{-1}{x^2} = \frac{-1}{x^2(1+e^x)},$$

provided $x > 0$.