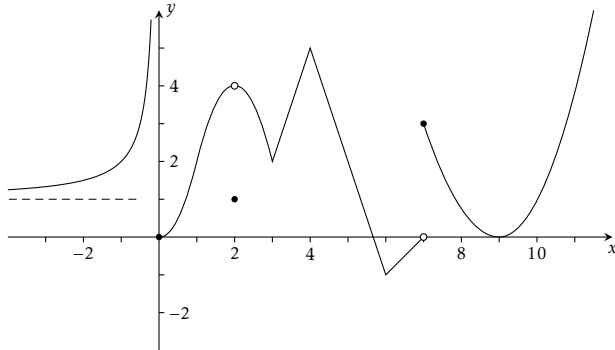


1. Below is the a sketch of the graph of a function f . (**Note:** This question is essentially asking you to bs.)



Determine each of the following.

a. $\lim_{x \rightarrow -\infty} f(x)$ b. $\lim_{x \rightarrow 0^-} (5 - f(x))$ c. $\lim_{x \rightarrow 7^-} \frac{1}{f(x)}$ d. $\lim_{h \rightarrow 0} \frac{f(5+h) - f(5)}{h}$

e. All values of x at which f is not differentiable.

2. Evaluate each of the following limits.

a. $\lim_{x \rightarrow 5} \frac{2\sqrt{x-1} - \sqrt{x^2-9}}{2x^2 - 9x - 5}$

b. $\lim_{x \rightarrow 3} \frac{9 - x^2}{x^2 - |6 - 5x|}$

c. $\lim_{x \rightarrow 0} \frac{\tan(\pi x)}{x}$

d. $\lim_{x \rightarrow 0^+} \sqrt{x + x^2} \cos\left(\frac{\pi}{x}\right)$

e. $\lim_{x \rightarrow 0} \frac{x - \cos(x)}{x^2}$

3. List all asymptotes of the curve defined by $y = \frac{1 + e^x}{e^x - 5}$.

4. Find all values of a , if any, so that the function f is continuous on \mathbb{R} .

$$f(x) = \begin{cases} ax^2 - 5 & \text{if } x < 2, \\ a^2 & \text{if } x = 2 \text{ and} \\ x^2 + ax + 7 & \text{if } x > 2. \end{cases}$$

5. State the definition of the derivative and use it to compute $f'(x)$, where

$$f(x) = \frac{1}{2x^2 + 5}.$$

6. Find $\frac{dy}{dx}$ for each of the following. Do not simplify your answers.

a. $y = \sqrt[3]{x^5} - \frac{3^x}{\log(5x)} + 6e^{7\pi} - \sec\left(\frac{8}{x}\right)$ b. $\sin(x^2 y) = ye^x$

c. $y = \frac{\tan^2(x)}{(8x^2 - 7)\sqrt{5x + 1}}$ d. $y = (e^x(x^2 - 5) + \log(x))^9$

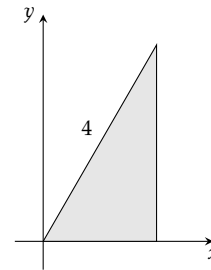
e. $y = \left(\frac{\csc(x)}{3}\right)^{\log_2(x)}$

7. Find an expression for $f^{(57)}(x)$, given that $f(x) = xe^x + 10x^{32}$.

8. Find all points on the graph of $y = \log(x^3 - 3x^2 - 9x + 10)$, at which the tangent line is horizontal.

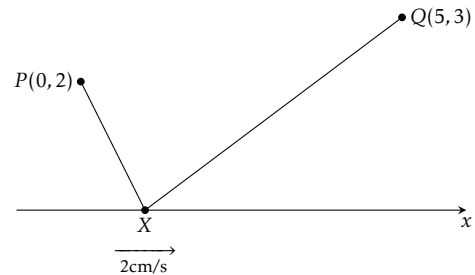
9. Sketch the graph of $y = x^4 + 4x^3$. Be sure that your solution includes all intercepts, intervals of monotonicity and concavity, and all extreme values and points of inflection.

10. A line segment of length 4 rotates counter-clockwise about the origin from the positive x axis to the positive y axis, and a triangle is formed by connecting the tip of the line segment to the x axis perpendicularly (as in the figure below). What is the maximum area of the triangle?



11. Find the extreme values of $f(x) = (2x)^{2/3}(10 - x) + 1$ on $\left[-4, \frac{1}{2}\right]$.

12. In the figure below, X is moving along the x -axis towards the right at a rate of 2cm/s. Find the rate at which $|PX| + |QX|$, the sum of the distances between X and the points $P(0,2)$ and $Q(5,3)$, is changing as X passes the point $(3,0)$.



13. Use the mean value theorem to show that $\sqrt[3]{1+x} < 1 + \frac{1}{3}x$ if $x > 0$.

14. The position of an object is given by $x = \sin^2(t) - \sin(t)$, for $t \geq 0$.

a. Express the velocity of the object as a function of time t .

b. Find the distance travelled by the object while $\pi \leq t < 2\pi$.

15. Evaluate each of the following integrals.

a. $\int \left(\frac{2}{3x^2} - \frac{4}{x} + 6^x - \pi^2 \right) dx$

b. $\int \frac{(2 - \sqrt{x})^2}{2\sqrt{x}} dx$

c. $\int_{\frac{1}{3}\pi}^{\frac{1}{4}\pi} \frac{\tan(x)}{\sec(x)} dx$

d. $\int_{-4}^4 (|x| - \sqrt{16 - x^2}) dx$

16. Given that $f''(x) = \frac{4}{x^{2/3}} - \frac{3}{x^{1/2}}$, $f'(1) = 12$ and $f(1) = 15$, find $f(x)$.

17. Compute $\int_0^2 (4x - 3x^2) dx$ as a limit of Riemann sums.

18. Compute and simplify $f'(x)$, where $f(x) = \int_1^{1/x} \frac{dt}{e^{1/t} + 1}$.

1. Nothing precise can be said about a function from a picture of its graph as given, although certain guesses are at least plausible.

a. It is plausible that the curve approaches the line defined by $y = 1$ as $x \rightarrow -\infty$, and that the limit is 1.

b. One might guess that $f(x) \rightarrow \infty$ as $x \rightarrow 0^-$, so that the limit is $-\infty$.

c. One might presume that $f(x) \rightarrow 0^-$ as $x \rightarrow 7^-$, so that the limit is $-\infty$.

d. The graph of f might be a straight line where $4 \leq x \leq 6$, and the slope might be roughly -3 (the ordinate appears to drop by 6, as the abscissa appears to increase by 2). So the limit could be -3 .

e. The graph appears to be discontinuous where $x = 0, 2$ and 7 , where f would not be differentiable. It may be that f is not differentiable at $3, 4$ and 6 , although it is easy to give functions whose graphs look just like those parts of f and are differentiable at those numbers.

2. a. If $a = 2\sqrt{x-1}$ and $b = \sqrt{x^2-9}$, then $a^2 - b^2 = 5 + 4x - x^2 = (5-x)(1+x)$. As, $2x^2 - 9x - 5 = (x-5)(2x+1)$, the factorization of a difference of squares gives

$$\lim_{x \rightarrow 5} \frac{a-b}{2x^2-9x-5} = \lim_{x \rightarrow 5} \frac{-(x+1)}{(a+b)(2x+1)} = \frac{-6}{(4+4)(11)} = -\frac{3}{44}.$$

b. If $x > \frac{6}{5}$ then $x^2 - |6-5x| = 6-5x+x^2 = (2-x)(3-x)$. Therefore, since $9-x^2 = (3-x)(3+x)$, it follows that

$$\lim_{x \rightarrow 3} \frac{9-x^2}{x^2-|6-5x|} = \lim_{x \rightarrow 3} \frac{3+x}{2-x} = -6.$$

c. Since $\sin(\vartheta)/\vartheta \rightarrow 1$ and $\cos(\vartheta) \rightarrow 1$ as $\vartheta \rightarrow 0$, it follows that

$$\lim_{x \rightarrow 0} \frac{\tan(\pi x)}{x} = \pi \lim_{x \rightarrow 0} \left\{ \frac{\sin(\pi x)}{\pi x} \cdot \frac{1}{\cos(\pi x)} \right\} = \pi.$$

d. If $0 < x < 1$ then $-\sqrt{2}\sqrt{x} \leq \sqrt{x^2 + \sqrt{x} \cos(\pi/x)} \leq \sqrt{2}\sqrt{x}$, so plainly

$$\lim_{x \rightarrow 0^+} \sqrt{x^2 + \sqrt{x} \cos\left(\frac{\pi}{x}\right)} = 0.$$

(The absolute value of the expression is $< \varepsilon$ if $0 < x < 1$ and $0 < 4x < \varepsilon^4$.)

e. Since $x - \cos(x) \rightarrow -1$ and $x^2 \rightarrow 0^+$ as $x \rightarrow 0$, it follows that

$$\lim_{x \rightarrow 0} \frac{x - \cos(x)}{x^2} = -\infty.$$

3. As $x \rightarrow \log(5)^\pm$, $1 + e^x \rightarrow 6$ and $e^x - 5 \rightarrow 0^\pm$, so $y \rightarrow \pm\infty$. Also,

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \frac{e^{-x} + 1}{1 - 5e^{-x}} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} y = \lim_{x \rightarrow -\infty} \frac{1 + e^x}{e^x - 5} = -\frac{1}{5}.$$

So the asymptotes of the curve are defined by $x = \log(5)$, $y = 1$ and $y = -\frac{1}{5}$.

4. The equation

$$f(2) = \lim_{x \rightarrow 2^-} f(x) \quad \text{is equivalent to} \quad a^2 = 4a - 5,$$

or $(a-2)^2 = -1$. So there is no real value of a for which f is continuous from the left at 2, let alone continuous everywhere.

5. The derivative of y with respect to x is defined by

$$\frac{dy}{dx} = \lim_{x' \rightarrow x} \frac{y' - y}{x' - x},$$

with its conventional domain (the set of all values of x for which the limit is defined).

If $z = 2x^2 + 5$ and $y = f(x) = 1/z$, then $z' - z = 2(x'^2 - x^2) = 2(x' - x)(x' + x)$ and $y' - y = (z - z')/(z'z) = -2(x' - x)(x' + x)/(z'z)$. Therefore,

$$f'(x) = \frac{dy}{dx} = \lim_{x' \rightarrow x} \frac{y' - y}{x' - x} = - \lim_{x' \rightarrow x} \frac{2(x' + x)}{z'z} = -\frac{4x}{z^2} = -\frac{4x}{(2x^2 + 5)^2}.$$

6. a. If $y = x^{5/3} - 3^x/\log(5x) + 6e^{7\pi} - 1/\cos(8/x)$, then

$$\frac{dy}{dx} = \frac{5}{3}x^{2/3} - \frac{3^x \log(3)}{\log(5x)} + \frac{3^x}{x(\log(5x))^2} + \frac{8 \sin(8/x)}{x^2 \cos^2(8/x)}.$$

b. If $\sin(x^2y) - ye^x = 0$, then implicit differentiation gives

$$\frac{dy}{dx} = -\frac{2xy \cos(x^2y) - ye^x}{x^2 \cos(x^2y) - e^x}.$$

c. If $y = \tan^2(x)(8x^2-7)^{-1}(5x+1)^{-1/2}$ then logarithmic differentiation gives

$$\frac{dy}{dx} = y \frac{d}{dx} \{\log|y|\} = \frac{\tan^2(x)}{(8x^2-7)\sqrt{5x+1}} \left\{ \frac{2}{\sin(x)\cos(x)} - \frac{16x}{8x^2-7} - \frac{5}{2(5x+1)} \right\}.$$

d. If $y = (e^x(x^2-5) + \log(x))^9$, then

$$\frac{dy}{dx} = 9(e^x(x^2-5) + \log(x))^8 (e^x(x^2+2x-5) + 1/x).$$

e. If $y = (3 \sin x)^{-\log_2(x)} = (3 \sin x)^{-\frac{\log(x)}{\log(2)}}$, then logarithmic differentiation gives

$$\frac{dy}{dx} = y \frac{d}{dx} \{\log(y)\} = -\frac{(3 \sin x)^{-\log_2(x)}}{\log(2)} \cdot \left\{ \frac{\log(3 \sin x)}{x} + \frac{\log(x) \cos(x)}{\sin(x)} \right\}.$$

7. Since $\frac{d}{dx} \{(x+a)e^x\} = (x+a+1)e^x$, if $a \in \mathbb{R}$ and $\frac{d^n}{dx^n} (x^{32}) = 0$ if $n > 32$, it follows that

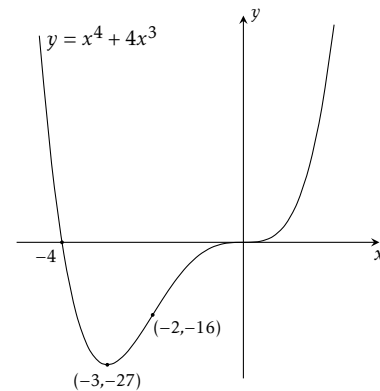
$$\frac{d^{57}}{dx^{57}} (xe^x + 10x^{32}) = e^x(x+57).$$

8. If $y = \log(x^3 - 3x^2 - 9x + 10)$, then

$$\frac{dy}{dx} = \frac{3x^2 - 6x - 9}{x^3 - 3x^2 - 9x + 10}, \quad \text{provided} \quad x^3 - 3x^2 - 9x + 10 > 0.$$

The numerator of the derivative is $3(x+1)(x-3)$. Since 3 does not belong to the domain of the function, the tangent line to the curve horizontal at the point $(-1, \log(15))$, and nowhere else.

9. If $f(x) = x^4 + 4x^3 = x^3(x+4)$, then $f'(x) = 4x^3 + 12x^2 = 4x^2(x+3)$ and $f''(x) = 12x^2 + 24x = 12x(x+2)$. The derivative $f'(x)$ is negative if $x < -3$ and is positive or zero if $x > -3$, so f is decreasing on $(-\infty, -3]$, increasing on $[-3, \infty)$ and has a local (and global) minimum value at the point $(-3, -27)$. The second derivative $f''(x)$ is positive if $x < -2$ or else $x > 0$ and is negative if $-2 < x < 0$, so the graph of f is concave up on $(-\infty, -2]$ and on $[0, \infty)$, is concave down on $[-2, 0]$, and has points of inflection at the origin and at the point $(-2, -16)$. The axis intercepts of the curve are the origin and $(-4, 0)$, and $f(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$. Below is a sketch of the curve (not to scale, the y axis is contracted by a factor of 10), with the points of interest (apart from the origin) labelled and emphasized.



10. If ϑ denotes the angle between the segment and the positive x axis, then the area of the triangle is $\frac{1}{2} \cdot 4 \cos(\vartheta) \cdot 4 \sin(\vartheta) = 4 \sin(2\vartheta)$, for $0 < \vartheta < \frac{1}{2}\pi$, and its largest value is 4, which occurs where $\vartheta = \frac{1}{4}\pi$. (i.e., where $\sin(2\vartheta) = 1$).

11. If $f(x) = (2x)^{2/3}(10-x) + 1$ then

$$f'(x) = \frac{4}{3}(2x)^{-1/3}(10-x) - (2x)^{2/3} = \frac{4(10-x) - 3(2x)}{3(2x)^{1/3}} = \frac{10(4-x)}{3(2x)^{1/3}},$$

so zero is the only critical number of f in $(-4, \frac{1}{2})$. Since $f(-4) = 57$, $f(0) = 1$ and $f(\frac{1}{2}) = \frac{21}{2}$, the extreme values of f on $[-4, \frac{1}{2}]$ are 1 and 57.

12. The sum of the distances is $s = |PX| + |QX| = \sqrt{x^2 + 4} + \sqrt{(x-5)^2 + 9}$ by Pythagoras' formula, where X is $(x, 0)$, so

$$\frac{ds}{dt} = \frac{2x}{\sqrt{x^2 + 4}} + \frac{2(x-5)}{\sqrt{(x-5)^2 + 9}}, \quad \text{since} \quad \frac{dx}{dt} = 2.$$

If x is 3 then $|PX| = |QX| = \sqrt{13}$, and thus $\frac{ds}{dt} = (2 \cdot 3 + 2(-2))\sqrt{13}^{-1} = \frac{2}{\sqrt{13}}\sqrt{13}$. Therefore, the sum $|PX| + |QX|$ is increasing at a rate of $\frac{2}{\sqrt{13}}\sqrt{13}$ cm/s as X passes the point $(3, 0)$.

13. If $f(x) = (1+x)^{1/3}$ then f is differentiable (and thus continuous) on $(-1, \infty)$. If $x > 0$, the mean value theorem applied to f on $[0, x]$ yields a real number ξ such that $0 < \xi < x$ and $f(x) = f(0) + f'(\xi)(x-0)$, or equivalently, $(1+x)^{1/3} = 1 + \frac{1}{3}x(1+\xi)^{-2/3}$. Since $\xi > 0$, it follows that $(1+\xi)^{-2/3} < 1$, and therefore $(1+x)^{1/3} < 1 + \frac{1}{3}x$, as required.

14. a. If the position of the object is $x = \sin^2(t) - \sin(t)$, then the velocity of the object is

$$\frac{dx}{dt} = 2\sin(t)\cos(t) - \cos(t) = (2\sin(t) - 1)\cos(t).$$

b. Since $\frac{dx}{dt} > 0$ if $\pi < t < \frac{3}{2}\pi$, the object travels $x(\frac{3}{2}\pi) - x(\pi) = 2$ units while $\pi \leq t \leq \frac{3}{2}\pi$. Since $\frac{dx}{dt} < 0$ if $\frac{3}{2}\pi < t < 2\pi$, the object travels $x(\frac{3}{2}\pi) - x(2\pi) = 2$ units while $\frac{3}{2}\pi \leq t \leq 2\pi$. So the object travels 4 units while $\pi \leq t \leq 2\pi$.

15. a. Integrating by inspection gives

$$\int \left(\frac{2}{3x^2} - \frac{4}{x} + 6^x - \pi^2 \right) dx = -\frac{2}{3x} - 4\log|x| + \frac{6^x}{\log(6)} - \pi^2 x + \alpha.$$

b. Since $(2 - \sqrt{x})^2 = 4 - 4\sqrt{x} + x$, it follows that

$$\int \frac{(2 - \sqrt{x})^2}{2\sqrt{x}} dx = \int \left(\frac{2}{\sqrt{x}} - 2 + \frac{1}{2}\sqrt{x} \right) dx = 4\sqrt{x} - 2x + \frac{1}{3}x\sqrt{x} + \beta.$$

c. Since $\tan(x)/\sec(x) = \sin(x)$, the integral is equal to

$$\int_{\frac{1}{3}\pi}^{\frac{1}{4}\pi} \sin(x) dx = -\cos(x) \Big|_{\frac{1}{3}\pi}^{\frac{1}{4}\pi} = \frac{1}{2} - \frac{1}{2}\sqrt{2}.$$

d. The definite integral of $|x|$ on $[-4, 4]$ is the area of a square of side 4, or 16, and the definite integral of $\sqrt{16-x^2}$ on $[-4, 4]$ is the area of a semicircle of radius 4, or 8π . Therefore,

$$\int_{-4}^4 (|x| - \sqrt{16-x^2}) dx = 16 - 8\pi.$$

16. If $f''(x) = 4x^{-2/3} - 3x^{-1/2}$ and $f'(1) = 12$ then, by inspection and the mean value theorem, $f'(x) = 12x^{1/3} - 6\sqrt{x} + 6$. Likewise, if $f(1) = 15$ then $f(x) = 9x^{4/3} - 4x^{3/2} + 6x + 4$, at least if $x > 0$ (also if $x = 0$, provided f is continuous from the right at 0).

17. If the interval $[0, 2]$ is divided into k subintervals of equal length, then the length of each subinterval is $2/k$, and the endpoints of the subintervals are $2j/k$, for $j = 0, 1, 2, \dots, k$. The Riemann sum obtained by evaluating the integrand at the right endpoint of each subinterval is

$$\begin{aligned} \mathcal{R}_k &= \frac{2}{k} \sum_{j=1}^k \left\{ 4 \cdot \frac{2j}{k} - 3 \left(\frac{2j}{k} \right)^2 \right\} = \frac{16}{k^2} \sum_{j=1}^k j - \frac{24}{k^3} \sum_{j=1}^k j^2 \\ &= \frac{16}{k^2} \cdot \frac{1}{2}k(k+1) - \frac{24}{k^3} \cdot \frac{1}{6}k(k+1)(2k+1) \\ &= 8 \left(1 + \frac{1}{k} \right) - 8 \left(1 + \frac{1}{k} \right) \left(1 + \frac{1}{2k} \right). \end{aligned}$$

Therefore,

$$\int_0^2 (4x - 3x^2) dx = \lim_{k \rightarrow \infty} \mathcal{R}_k = 8 - 8 = 0.$$

18. By the first fundamental theorem of calculus and the chain rule, if

$$f(x) = \int_1^{1/x} \frac{dt}{1+e^{1/t}} \quad \text{then} \quad f'(x) = \frac{1}{1+e^x} \cdot \frac{-1}{x^2} = \frac{-1}{x^2(1+e^x)},$$

provided $x > 0$.