1. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 4}\frac{3\sqrt{x}}{2x-5}$ b. $\displaystyle\lim_{x\to -3}\frac{x+3}{\sqrt{6-x}-3}$ c. $\displaystyle\lim_{x\to -3^-}\frac{4-x}{9-x^2}$

d. $\displaystyle\lim_{x\to -3}\frac{3x^2-|7x-6|}{2x^2+5x-3}$ e. $\displaystyle\lim_{x\to 1^+}\left\{4+\log(x)\sin\left(\frac{2}{x-1}\right)\right\}$

2. Find all values of a and b, if any, so that the function f, defined by

$$f(x)=\begin{cases}\dfrac{7+6x-x^2}{x+1} & \text{if } x<-1,\\ ax+b & \text{if } -1\leqslant x\leqslant 4 \text{ and}\\ 2^{5-x}+16 & \text{if } x>4,\end{cases}$$

is everywhere continuous.

3. Find $f'(x)$ using the limit definition of the derivative, where

$$f(x)=\frac{1}{\sqrt{2x-1}}.$$

4. Find $\frac{dy}{dx}$ for each of the following.

a. $y=6^x-2\sqrt[3]{x^5}+\csc(2x+1)+\dfrac{8}{x}$ b. $y=\dfrac{x^2+e^{2x}}{x+\sqrt{\tan(x^4)}}$

c. $y=(\log x)^{\sec(x)}$ d. $y=\log\left(\sqrt[4]{\dfrac{(x+1)^3}{(2x-1)\sin(x)}}\right)$ e. $\cos(y)=\log(xy)$

5. Write an equation of the line tangent to the graph of $y^2-4xy=12$ at the point $(-1,2)$.

6. The position of a particle along the y-axis at time t is given by

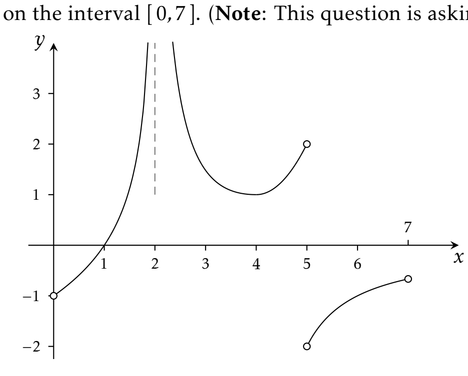
$$y=\frac{(t+2)^3}{t^2+1},\qquad \text{where}\quad t\geqslant 0.$$

When is the particle rising?

7. A funnel with the shape of an inverted right circular cones has height 20 cm and radius 5 cm at its top, and is filled with water. Water drains out of the bottom of the funnel at a rate of $4\,\text{cm}^3$/sec. At what rate is the depth of the water decreasing when the water is 10 cm deep?

8. Find the largest and smallest values of $\log(x^2+x+1)$, where $-1\leqslant x\leqslant 1$.

9. Below is a sketch of the graph of the derivative of a function f which is continuous on the interval $[0,7]$. (**Note**: This question is asking you to BS.)



a. Write the intervals of monotonicity and concavity of f.

b. Given that $f(0)=-2$, sketch a graph of f.

10. Sketch the graph of f, given that

$$f(x)=\frac{x-4}{\sqrt{x^2+8}},\quad f'(x)=\frac{4(x+2)}{(x^2+8)^{3/2}}\quad\text{and}\quad f''(x)=\frac{8(x+4)(1-x)}{(x^2+8)^{5/2}}.$$

Make sure that your solution includes all intercepts, asymptotes, intervals of monotonicity and concavity, extreme values and points of inflection.

11. The strength of a rectangular beam is jointly proportional to its width and the square of its depth. Find the depth and width of the strongest rectangular beam which can be cut from a circular log of radius 10 cm.

12. Find $f(x)$ given that $f''(x)=\pi\sin(x)+1$, $f'(\pi)=0$ and $f(0)=\pi$.

13. Evaluate the integral

$$\int_0^3 (8x-2x^3)\,dx$$

as a limit of Riemann sums.

14. Compute each of the following integrals.

a. $\displaystyle\int(4\sqrt[5]{x^3}+5e^x+\sqrt{\pi})\,dx$ b. $\displaystyle\int\frac{(\sqrt{x}+3)^2}{2x}\,dx$ c. $\displaystyle\int\sin(x)\sec^2(x)\,dx$

d. $\displaystyle\int_0^3|x^2-1|\,dx$ e. $\displaystyle\int_{-2}^4\left(|x-2|-\sqrt{16-x^2}\right)dx$

15. Given that the function f is continuous on the interval $[-1,3]$ and that

$$\int_{-1}^1 f=3,\quad \int_2^3 f=-2\quad\text{and}\quad\int_1^3 f=5,\qquad\text{evaluate}\quad\int_{-1}^2 f.$$

16. Given that

$$g(x)=\int_1^x f(t)\,dt\qquad\text{and}\qquad f(x)=\int_0^{x^2}\sqrt{t+9}\,dt,$$

evaluate $g(1)$ and $g''(4)$.

17. Show that if $f''(x)>0$ on the interval (a,b), then the graph of $y=e^{f(x)}$ is concave upwards on the interval (a,b).

1. a. Direct substitution gives $\dfrac{3\sqrt{4}}{2\cdot 4-5}=2$.

b. Rationalizing the denominator and simplifying gives

$$\lim_{x\to -3}\frac{x+3}{\sqrt{6-x}-3}=-\lim_{x\to -3}\left\{\sqrt{6-x}+3\right\}=-6,$$

by independence and direct substitution.

c. Since $4-x\to 7$ and $9-x^2\to 0^-$ as $x\to -3^-$, the limit $(-\infty)$ is undefined.

d. Since $2x^2+5x-3=(2x-1)(x+3)$, and $3x^2-|7x-6|=(3x-2)(x+3)$ if $x<\frac{6}{7}$, it follows that

$$\lim_{x\to -3}\frac{3x^2-|7x-6|}{2x^2+5x-3}=\lim_{x\to -3}\frac{3x-2}{2x-1}=\frac{11}{7},$$

by independence and direct substitution.

e. As $\log(x)\to 0$ as $x\to 1^+$ and $|\sin(2/(x-1))|\leqslant 1$ for $x\neq 1$, it is plain that

$$\lim_{x\to 1^+}\left\{4+\log(x)\sin\left(\frac{2}{x-1}\right)\right\}=4+0=4.$$

(For $|\log(x)\sin(2/(x-1))|<\varepsilon$ if $1<x<1+\varepsilon$ from the definition of $\log(x)$.)

2. Plainly the function f is everywhere continuous if, and only if, it is continuous at -1 and 4. Now $f(x)=7-x$ if $x<-1$, so

$$\lim_{x\to -1^-}f(x)=8,\qquad\text{and}\qquad f(-1)=\lim_{x\to -1^+}f(x)=-a+b,$$

so f is continuous at -1 if, and only if, $-a+b=8$. Next,

$$f(4)=\lim_{x\to 4^-}f(x)=4a+b\qquad\text{and}\qquad\lim_{x\to 4^+}f(x)=18,$$

so f is continuous at 4 if, and only if, $4a+b=18$. Combining the equations (in a and b) gives $5a=10$, or $a=2$, which yields $b=10$. Therefore, f is everywhere continuous if, and only if, $a=2$ and $b=10$.

3. If $y=\sqrt{2x-1}$ then $y-y'=(y^2-y'^2)/(y'+y)=-2(x'-x)/(y'+y)$, and if $z=1/y$ then $z'-z=(y-y')/(y'y)=-2(x'-x)/(y'y(y'+y))$. Therefore,

$$\frac{dz}{dx}=\lim_{x'\to x}\frac{z'-z}{x'-x}=\lim_{x'\to x}\frac{-2}{y'y(y'+y)}=\frac{-2}{2y^3}=-\frac{1}{(2x-1)^{3/2}}.$$

4. a. If $y=6^x-2\sqrt[3]{x^5}+\csc(2x+1)+8/x$, then

$$\frac{dy}{dx}=6^x\log(6)-\tfrac{10}{3}x^{2/3}-2\csc(2x+1)\cot(2x+1)-\frac{8}{x^2}.$$

b. If $y=\dfrac{x^2+e^{2x}}{x+\sqrt{\tan(x^4)}}$, then

$$\frac{dy}{dx}=\frac{2(x+e^{2x})}{x+\sqrt{\tan(x^4)}}-\frac{x^2+e^{2x}}{\left(x+\sqrt{\tan(x^4)}\right)^2}\cdot\left(1+\frac{2x^3\sec^2(x^4)}{\sqrt{\tan(x^4)}}\right).$$

c. If $y=(\log x)^{\sec(x)}$, then

$$\frac{dy}{dx}=y\frac{d}{dx}\left\{\log|y|\right\}=(\log x)^{\sec(x)}\cdot\left\{\frac{\sec(x)}{x\log(x)}+\frac{\log(\log x)\sin(x)}{\cos^2(x)}\right\}.$$

d. If $y=\log\left(\sqrt[4]{\dfrac{(x+1)^3}{(2x-1)\sin(x)}}\right)=\frac{3}{4}\log|x+1|-\frac{1}{4}\log|2x-1|-\frac{1}{4}\log|\sin(x)|$, then

$$\frac{dy}{dx}=\frac{3}{4(x+1)}-\frac{1}{2(2x-1)}-\frac{\cos(x)}{4\sin(x)}.$$

e. If $\cos(y)=\log(xy)$, equivalently, $\log(x)+\log(y)-\cos(y)=0$, then

$$\frac{dy}{dx}=-\frac{1/x}{1/y+\sin(y)}=-\frac{y}{x(y\sin(y)+1)}.$$

5. If $y^2-4xy=12$ then

$$\left.\frac{dy}{dx}\right|_{\substack{x=-1\\y=2}}=\left.\frac{2y}{y-2x}\right|_{\substack{x=-1\\y=2}}=1,\qquad\text{so}\qquad\frac{y-2}{x+1}=1,$$

or $y=x+3$, defines the tangent line to the curve at the point $(-1,2)$.

6. If $y=\dfrac{(t+2)^2}{t^2+1}$, for $t\geqslant 0$, then

$$\frac{dy}{dt}=\frac{3(t+2)^2}{t^2+1}-\frac{2t(t+2)^3}{(t^2+1)^2}=\frac{(t+2)^2(t-1)(t-3)}{(t^2+1)^2}>0$$

if, and only if, $0\leqslant t<1$, or else $t>3$; so the particle is rising at these times.

7. If the depth of water in the funnel is h(cm), then its volume is

$$V=\tfrac{1}{3}\pi r^2h=\tfrac{1}{3}\pi\left(\tfrac{1}{4}h\right)^2h=\tfrac{1}{48}\pi h^3,$$

since $h=4r$ by similarity. Differentiating with respect to time gives

$$\frac{dV}{dt}=\frac{1}{16}h^2\frac{dh}{dt},\qquad\text{so}\qquad -4=\frac{1}{16}\pi\cdot 10^2\left.\frac{dh}{dt}\right|_{h=10}=\frac{25}{4}\pi\left.\frac{dh}{dt}\right|_{h=10}.$$

Therefore, when the water is 10 centimetres deep, its depth is decreasing at a rate of $16/(25\pi)$ centimetres per second.

8. If $y=\log(x^2+x+1)$, then

$$\frac{dy}{dx}=\frac{2x+1}{x^2+x+1},$$

which is zero if, and only if, $x=-\frac{1}{2}$. Comparing

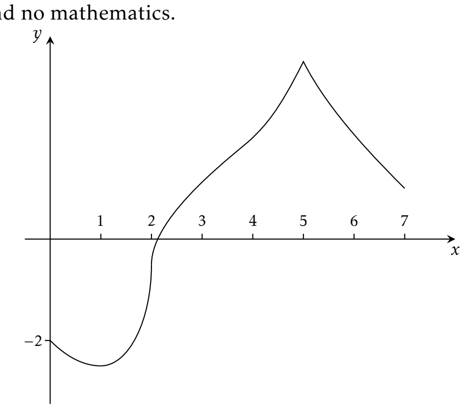
$$\left.y\right|_{x=-1}=\log(1)=0,\qquad\left.y\right|_{x=-\frac{1}{2}}=\log\left(\tfrac{3}{4}\right)\qquad\text{and}\qquad\left.y\right|_{x=1}=\log(3),$$

reveals that the smallest value of y on $[-1,1]$ is $\log(\frac{3}{4})$ and the largest value of y on $[-1,1]$ is $\log(3)$.

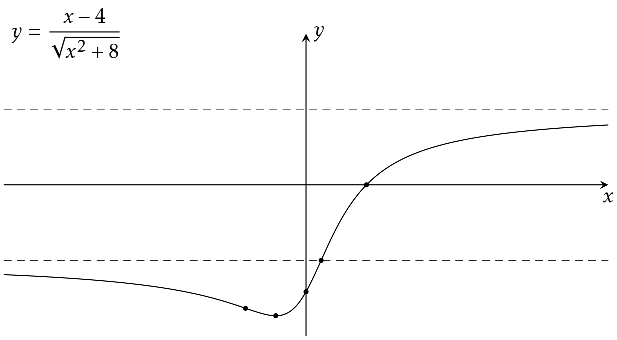
9. There are many answers to this question, each of which is BS, being based on unsound assumptions which are not implied by the given picture.

a. Making many assumptions which are not mathematically implied by the picture, one could guess that f is increasing on $[1,5]$, and decreasing on $[0,1]$ and on $[5,7]$. One could also guess that the graph of f is concave up on $[0,2]$ and on $[4,5]$, and that the graph of f is concave down on $[2,4]$.

b. Based on the guesses in part a, one could make a sketch such as the one below. On the other hand, there is a continuous function whose graph looks like a horizontal line, whose derivative has a graph which looks like the picture given in the question. So any sketch, such as the one below, involves lots of BS and no mathematics.



10. The domain of f is $\mathbb{R}$, and its graph has intercepts $(4,0)$ and $(0,-\sqrt{2})$, no vertical asymptotes, and horizontal asymptotes defined by $y=\pm 1$, since $f(x)\to\pm 1$ as $x\to\pm\infty$. The derivative $f'(x)$ is negative if $x<-2$ and positive if $x>-2$. So f is decreasing on $(-\infty,-2]$, increasing on $[-2,\infty)$ and has a local (and global) minimum at the point $(-2,-\sqrt{3})$. The second derivative $f''(x)$ is negative if $x<-4$ or $x>1$, and positive if $-4<x<1$. So the graph of f is concave down on $(-\infty,-4]$ and on $[1,\infty)$, **but not on the union of these intervals**, and concave up on $[-4,1]$, with points of inflection at $\left(-4,-\frac{2}{3}\sqrt{6}\right)$ and $(1,-1)$. Below is a sketch of the graph of f (not to scale—the x axis is contracted by a factor of 5), with the asymptotes drawn as dashed lines and the points of interest emphasized.



11. If w and h denote, respectively, the width and depth of the beam, each measured in centimetres, then the beam's strength is a positive multiple of $wh^2=w(100-w^2)=100w-w^3$ (by Pythagoras' formula). So the derivative of the strength of the beam is a positive multiple of $100-3w^2$, which is positive if $0<w<\frac{10}{3}\sqrt{3}$ and negative if $\frac{10}{3}\sqrt{3}<w$. Therefore, the strongest beam is $\frac{10}{3}\sqrt{3}$ cm wide and $\frac{10}{3}\sqrt{6}$ cm deep.

12. If $f''(x)=\pi\sin(x)+1$ and $f'(\pi)=0$, then $f'(x)=x-2\pi-\pi\cos(x)$ by the mean value theorem. Likewise, if $f(0)=\pi$ then $f(x)=\frac{1}{2}x^2-2\pi x+\pi-\pi\sin(x)$.

13. If the interval $[0,3]$ is divided into k subintervals of equal length, then the length of each subinterval is $3/k$, the endpoints of the subintervals are $3j/k$ and the values of the integrand at the endpoints are $24j/k-54j^3/k^3$, for $j=0,1,2,\dots,k$. The Riemann sum obtained by evaluating the integrand at the right endpoint of each subinterval is

$$\mathscr{R}_k=\frac{3}{k}\sum_{j=1}^{k}\left\{\frac{24}{k}j-\frac{54}{k^3}j^3\right\}=\frac{72}{k^2}\sum_{j=1}^{k}j-\frac{162}{k^4}\sum_{j=1}^{k}j^3$$
$$=\frac{72}{k^2}\cdot\frac{1}{2}k(k+1)-\frac{162}{k^4}\cdot\frac{1}{4}k^2(k+1)^2=36\left(1+\frac{1}{k}\right)-\frac{81}{2}\left(1+\frac{1}{k}\right)^2.$$

Therefore

$$\int_0^3(8x-2x^3)\,dx=\lim_{k\to\infty}\mathscr{R}_k=36-\frac{81}{2}=-\frac{9}{2}.$$

14. a. By inspection,

$$\int\left(4\sqrt[5]{x^3}+5e^x+\sqrt{\pi}\right)dx=\frac{5}{2}x^{8/5}+5e^x+\sqrt{\pi}\,x+\alpha.$$

b. Since $\frac{1}{2}x^{-1}(\sqrt{x}+3)^2=\frac{1}{2}+3x^{-1/2}+\frac{9}{2}x^{-1}$ for $x>0$, it follows that

$$\int\frac{(\sqrt{x}+3)^2}{2x}\,dx=\frac{1}{2}x+6\sqrt{x}+\frac{9}{2}\log(x)+\beta.$$

c. Revising the integrand and integrating by inspection gives

$$\int\sin(x)\sec^2(x)\,dx=\int\frac{\sin(x)}{\cos^2(x)}\,dx=\frac{1}{\cos(x)}+\gamma.$$

d. Basic properties of the definite integral yield

$$\int_0^3|x^2-1|\,dx=\int_1^0(x^2-1)\,dx+\int_1^3(x^2-1)\,dx=\left(\tfrac{1}{3}x^3-x\right)\Big|_1^0+\left(\tfrac{1}{3}x^3-x\right)\Big|_1^3$$
$$=\frac{22}{3}.$$

e. The definite integral of $|x-2|$ on the interval $[-2,4]$ is half the sum of the area of a square of side 4 and the area of a square of side 2, or $8+2=10$. The definite integral of $\sqrt{16-x^2}$ on the interval $[-2,4]$ is equal to the sum of the area of a sector of radius 4 and angle $\frac{2}{3}\pi$, and the area of a triangle of base 2 and height $2\sqrt{3}$, or $\frac{1}{2}\cdot\frac{2}{3}\pi\cdot 4^2+\frac{1}{2}\cdot 2\cdot 2\sqrt{3}=\frac{16}{3}\pi+2\sqrt{3}$. The definite integral in question is the difference between these areas, so

$$\int_{-2}^{4}\left(|x-2|-\sqrt{16-x^2}\right)dx=10-2\sqrt{3}-\frac{16}{3}\pi.$$

15. By the interval additivity of the definite integral

$$\int_{-1}^{2}f=\int_{-1}^{1}f+\int_{1}^{3}f-\int_{3}^{2}f=3+5-(-2)=10.$$

16. If

$$g(x)=\int_1^x f\qquad\text{and}\qquad f(x)=\int_0^{x^2}\sqrt{y+9}\,dy,$$

then f is differentiable (and thus also continuous) on $\mathbb{R}$, so $g(1)=\int_1^1 f=0$,

$$g'(x)=f(x)=\int_0^{x^2}\sqrt{y+9}\,dy\quad\text{and}\quad g''(4)=f'(4)=\sqrt{4^2+9}\cdot 2\cdot 4=40,$$

by the first fundamental theorem of calculus and the chain rule.

17. For $a<x<b$, it is given that $f''(x)>0$, and so

$$\frac{d^2}{dx^2}\left(e^{f(x)}\right)=\frac{d}{dx}\left(e^{f(x)}f'(x)\right)=e^{f(x)}\left((f'(x))^2+f''(x)\right)>0,$$

since $e^{f(x)}>0$ and $(f'(x))^2\geqslant 0$. Hence, the graph of $y=e^{f(x)}$ is concave upward on (a,b).