1. Evaluate each of the following limits.

a. $\lim_{x\to 2} \frac{3x^2-11x+10}{x^3-8}$ b. $\lim_{x\to 0} \frac{x+\sin(5x)}{\sin(2x)}$ c. $\lim_{x\to \frac{1}{3}\pi^+} \frac{1}{2-\sec(x)}$

d. $\lim_{x\to\infty} \left\{x-\sqrt{x^2+5x}\right\}$ e. $\lim_{x\to 5^+} \frac{25-x^2}{|x-6|-1}$

2. What value of c makes the following function continuous at 2?

$$f(x) = \begin{cases} c^2x+3c & \text{if } x<2, \\ x & \text{if } x=2 \text{ and} \\ cx+c^2+2 & \text{if } x>2. \end{cases}$$

3. Write an equation of each asymptote of the function f, defined by

$$f(x) = \frac{x+\sin(x)}{3x+2}.$$

4. Use the definition of the derivative to compute $\frac{d}{dx}\left\{\frac{1}{3x+2}\right\}$.

5. Find $\frac{dy}{dx}$ for each of the following.

a. $y = \frac{8}{x} - \sqrt[3]{x} + 2^x$ b. $y = \sqrt{\frac{x^2-1}{x^2+1}}$

c. $y = \cos^3(6x^2)$ d. $\log(x-y) = xy-2$

6. Compute $f'(0)$ using logarithmic differentiation, where

$$f(x) = \frac{2^{x+3}\sqrt{9-x^2}}{(x+1)^4(6x+3)}.$$

Simplify your answer as much as possible.

7. Determine the number of real solutions of the equation $x^3+2=4x$.

8. Find all points P on the parabola defined by $y=x^2-2x$ for which the line through P and the point $(4,4)$ is tangent to the parabola at P.

9. a. State the mean value theorem.

b. Show that if $f(2)=-2$ and $f'(x)\geqslant 5$ for $x\geqslant 2$ then $f(4)\geqslant 8$.

10. Write an equation of the line tangent to the curve defined by

$$x^3+y^3 = 8(xy+1)$$

at the point $(-1,1)$.

11. A lighthouse is located on a small island 2 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How quickly is the beam moving along the shoreline when it is $\frac{1}{2}$ km from P?

12. What is the largest possible volume of a right circular cylinder which is inscribed in a sphere of radius 3?

13. The position of an object moving along a straight line at time $t\geqslant 0$ is given by $s=(t-3)^2e^{-t}$. When is the object at rest? When is the object moving in the positive direction.

14. Find the absolute extrema of $f(x) = \frac{x+18}{\sqrt{x^2+36}}$ on $[0,8]$.

15. Sketch the graph of f, given that

$$f(x) = \frac{8(x^2+4)}{(x+2)^2}, \quad f'(x) = \frac{32(x-2)}{(x+2)^3} \quad \text{and} \quad f''(x) = \frac{64(4-x)}{(x+2)^4}.$$

Make sure that your solution includes all intercepts, asymptotes, intervals of monotonicity and concavity, extreme values and points of inflection.

16. Evaluate each of the following integrals.

a. $\int\left(e^2-\frac{4}{x}+\sqrt[3]{x^5}\right)dx$ b. $\int\frac{(x^5+1)^2}{x^4}\,dx$

c. $\int\sec(x)(\sec(x)+\tan(x))\,dx$ d. $\int_0^{\frac{1}{2}\pi}\left|\frac{1}{2}-\sin(x)\right|dx$

17. Find the derivative with respect to x of $y = \int_0^{x^2}\frac{t}{1+t^2}\,dt$.

18. Evaluate the definite integral

$$\int_0^2 (2x^2+1)\,dx$$

as a limit of Riemann sums.

19. Evaluate $\int_{-2}^{2}(x-2)f(x)\,dx$, given that f is even, $\int_0^3 f = 216$ and $\int_{-2}^3 f = 240$.

1. a. Since $3x^2 - 11x + 10 = (x-2)(3x-5)$ and $x^2 - 8 = (x-2)(x^2+2x+4)$, the limit is equal to

$$\lim_{x\to 2} \frac{3x-5}{x^2+2x+4} = \tfrac{1}{12}.$$

b. Revising the expression in the limit gives

$$\lim_{x\to 0}\left\{\frac{1}{2}\cdot\frac{2x}{\sin(2x)} + \frac{5}{2}\cdot\frac{\sin(5x)}{5x}\cdot\frac{2x}{\sin(2x)}\right\} = 3,$$

since $\displaystyle\lim_{\vartheta\to 0}\frac{\sin(\vartheta)}{\vartheta} = 1$.

c. Since $\sec(x) \to 2^+$ as $x \to \frac{1}{3}\pi^+$, the limit $(-\infty)$ is undefined.

d. If $x > 0$ then

$$y = x - \sqrt{x^2+5x} = \frac{-5x}{x+\sqrt{x^2+5x}} = \frac{-5}{1+\sqrt{1+5/x}},$$

so $y \to -\frac{5}{2}$ as $x \to \infty$ (by inspecting dominant terms).

e. If $x < 6$ then $|x-6| - 1 = 5 - x$, and in any case $25 - x^2 = (5-x)(5+x)$, so the limit is equal to

$$\lim_{x\to 5^+}(5+x) = 10.$$

2. Since

$$\lim_{x\to 2^-} f(x) = 2c^2 + 3c, \quad f(2) = 2 \quad\text{and}\quad \lim_{x\to 2^+} f(x) = c^2 + 2c + 2,$$

the function f is continuous at 2 if, and only if, $2c^2+3c = 2$ and $c^2+2c+2 = 2$; equivalently, $(2c-1)(c+2) = 0$ and $c(c+2) = 0$. Therefore, the function f is continuous at 2 if, and only if, c is -2.

3. As $x \to -\frac{2}{3}^{\pm}$, $3x+2 \to 0^{\pm}$ and $x + \sin(x) \to -\frac{2}{3} - \sin\left(\frac{2}{3}\right) < 0$, so $f(x) \to \mp\infty$. Since f is continuous at all real numbers besides $-\frac{2}{3}$, the graph of f has one vertical asymptote, defined by $x = -\frac{2}{3}$. Next,

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{1+(\sin x)/x}{3+\frac{2}{x}} = \tfrac{1}{3},$$

since $0 \leqslant |(\sin x)/x| \leqslant |1/x| < |2/x| \to 0$ as $x \to \pm\infty$. Thus, the graph of f has one horizontal asymptote, defined by $y = \frac{1}{3}$.

4. Since

$$\frac{1}{3x'+2} - \frac{1}{3x+2} = \frac{-3(x'-x)}{(3x'+2)(3x+2)},$$

it follows that

$$\frac{d}{dx}\left\{\frac{1}{3x+2}\right\} = \lim_{x'\to x}\frac{-3}{(3x'+2)(3x+2)} = -\frac{3}{(3x+2)^2}.$$

5. a. If $y = \dfrac{8}{x} - \sqrt[3]{x} + 2^x$, then

$$\frac{dy}{dx} = -\frac{8}{x^2} - \tfrac{1}{3}x^{-2/3} + 2^x\log(2).$$

b. If $y = \sqrt{\dfrac{x^2-1}{x^2+1}} = \sqrt{1 - \dfrac{2}{x^2+1}}$, then

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x^2+1}{x^2-1}}\cdot\frac{2\cdot 2x}{(x^2+1)^2} = \frac{2x}{\sqrt{(x^2-1)(x^2+1)^3}}.$$

c. If $y = \cos^3(6x^2)$, then $\dfrac{dy}{dx} = -36x\cos^2(6x^2)\sin(6x^2)$.

d. If $\log(x-y) = xy - 2$, equivalently $xy - \log(x-y) = 2$, then

$$\frac{dy}{dx} = -\frac{y - 1/(x-y)}{x + 1/(x-y)} = \frac{1 - y(x-y)}{1 + x(x-y)}.$$

6. Logarithmic differentiation gives

$$f'(0) = f(0)\frac{d}{dx}\left\{\log|f(x)|\right\}$$

$$= f(x)\frac{d}{dx}\left\{(x+3)\log(2) + \tfrac{1}{2}\log(9-x^2) - 4\log|x+1| - \log|3(2x+1)|\right\}\bigg|_{x=0}$$

$$= \frac{2^3\sqrt{9}}{1^4\cdot 3}\left\{\frac{\log(2)}{x+3} - \frac{x}{9-x^2} - \frac{4}{x+1} - \frac{2}{2x+1}\right\}\bigg|_{x=0} = 8\left(\log(2) - 0 - 4 - 2\right)$$

$$= 8\log(2e^{-6}).$$

7. Solutions of the equation are zeros of the polynomial $p(x) = x^3 - 4x + 2$, so the equation has at most three real solutions (the degree of p). Since

$$p(-3) = -13, \qquad p(\pm 2) = p(0) = 2 \qquad\text{and}\qquad p(1) = -1,$$

and p is continuous on $\mathbb{R}$ (being a polynomial function), the intermediate value theorem implies that p has a zero in each of the intervals $(-3,-2)$, $(0,1)$ and $(1,2)$. Therefore, the equation in question has three real solutions.

8. The tangent to the parabola at the point $P(x, x^2-2x)$ contains the point $(4,4)$ if, and only if,

$$\frac{x^2-2x-4}{x-4} = 2x - 2, \qquad\text{or}\qquad x^2 - 2x - 4 = 2x^2 - 10x + 8.$$

Equivalently, $0 = x^2 - 8x + 12 = (x-2)(x-6)$. So the tangent lines to the parabola at the points $(2,0)$ and $(6,24)$ pass through the point $(4,4)$, and these are the only such points on the parabola.

9. a. If $a < b$, and f is continuous on $[a,b]$ and differentiable on (a,b), then there is a real number ξ such that $a < \xi < b$ and $f(b) = f(a) + f'(\xi)(b-a)$.

b. Since f is differentiable, and therefore also continuous, on $[2,\infty)$, the mean value theorem applies to f on $[2,4]$, and yields a real number ξ such that $2 < \xi < 4$ and

$$f(4) = f(2) + f'(\xi)(4-2) = -2 + 2f'(\xi) \geqslant -2 + 2\cdot 5 = 8,$$

since it is given that $f' \geqslant 5$ on $[2,\infty)$.

10. The given equation is equivalent to $x^3 + y^3 - 8xy = 8$, so

$$\frac{dy}{dx}\bigg|_{\substack{x=-1\\ y=\ 1}} = -\frac{3x^2-8y}{3y^2-8x}\bigg|_{\substack{x=-1\\ y=\ 1}} = \tfrac{5}{11}, \qquad\text{and thus}\qquad \frac{y-1}{x+1} = \tfrac{5}{11},$$

or $5x - 11y = -16$, defines the line tangent to the given curve at $(-1,1)$.

11. If ϑ denotes the angle between the beam and the segment joining the lighthouse to P and $|x|$ denotes the distance between the point where the beam meets the shoreline and P, then $x = 2\tan(\vartheta)$, at least if $-\frac{1}{2}\pi < \vartheta < \frac{1}{2}\pi$, and $\frac{d\vartheta}{dt} = \pm 4\cdot 2\pi = \pm 8\pi$ radians per minute (depending on whether the beam is moving away from, or towards, P). Thus, since $\tan(\vartheta) = \frac{1}{4}$ if $x = \pm\frac{1}{2}$,

$$\frac{dx}{dt}\bigg|_{x=\pm\frac{1}{2}} = 2\left(1 + \tan^2(\vartheta)\right)\frac{d\vartheta}{dt}\bigg|_{x=\pm\frac{1}{2}} = 2\left(1 + \tfrac{1}{16}\right)\cdot(\pm 8\pi) = \pm 17\pi.$$

So the beam is moving along the shoreline at a rate of 17π km/min when it is $\frac{1}{2}$ km from P.

12. If x denotes the radius of the cylinder and $2y$ denotes the height of the cylinder, then $x^2 + y^2 = 9$ and the volume of the cylinder is

$$2\pi x^2 y = 2\pi(9-y^2)y = 2\pi(9y - y^3).$$

for $0 \leqslant y \leqslant 3$. Since

$$\frac{d}{dy}\left\{2\pi(9y-y^3)\right\} = 2\pi(9-3y^2) = 6\pi(3-y^2),$$

the only critical number of the volume in $(0,3)$ is $\sqrt{3}$, at which the volume is largest (for the volume is zero if y is 0 or 3). The largest possible volume of such a cylinder is thus $2\pi(9-3)\sqrt{3} = 12\sqrt{3}\,\pi$.

13. If $s = (t-3)^2e^{-t}$, then

$$\frac{ds}{dt} = 2(t-3)e^{-t} - (t-3)^2e^{-t} = e^{-t}(t-3)(5-t).$$

Thus, the object is at rest when $t = 3$ or $t = 5$, and the object is moving in the positive direction while $3 < t < 5$.

14. If $f(x) = \dfrac{x+18}{\sqrt{x^2+36}}$, then

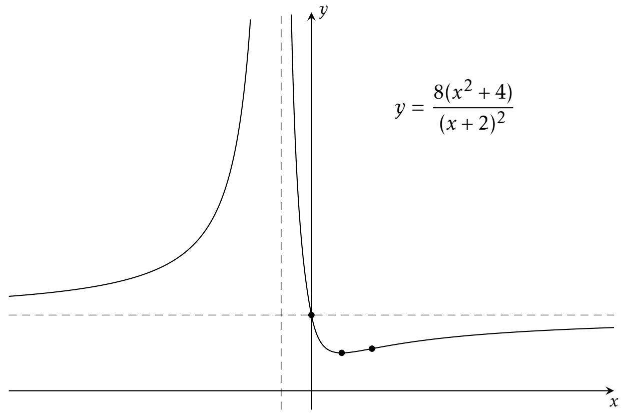
$$f'(x) = \frac{1}{\sqrt{x^2+36}} - \frac{x(x+18)}{(x^2+36)^{3/2}} = \frac{18(2-x)}{(x^2+36)^{3/2}},$$

so the only critical number of f in $(0,8)$ is 2. Comparing

$$f(0) = 3, \qquad f(2) = \frac{20}{\sqrt{40}} = \sqrt{10} \qquad\text{and}\qquad f(8) = \frac{26}{\sqrt{100}} = \tfrac{13}{5},$$

reveals that the largest and smallest values of f on $[0,8]$ are, respectively, $\sqrt{10}$ and $\frac{13}{5}$.

15. The domain of f is $\mathbb{R} \setminus \{-2\}$, the graph has a vertical asymptote defined by $x = -2$, a horizontal asymptote defined by $y = 8$, no x-intercept and its y-intercept is $(0,8)$, where the curve meets its horizontal asymptote. The derivative $f'(x)$ positive if $x < -2$ or $x > 2$ and negative if $-2 < x < 2$. So f is increasing on $(-\infty,-2)$ and on $[2,\infty)$, f is decreasing on $(-2,2]$, and f has one local minimum at the point $(2,4)$. The second derivative $f''(x)$ is positive if $x < -2$ or else $-2 < x < 4$ and negative if $x > 4$. So the graph of f is concave up on $(-\infty,-2)$ and on $(-2,4]$, the graph is concave down on $[4,\infty)$, and the graph has one point of inflection at $\left(4, \frac{40}{9}\right)$. Below is a sketch of the graph of f (not to scale, the y axis is contracted by a factor of $\frac{5}{8}$), with the asymptotes drawn as dashed lines and the points of interest emphasized.



16. a. Integrating by inspection gives

$$\int \left(e^2 - \frac{4}{x} + \sqrt[3]{x^5} \right) dx = e^2 x - 4 \log|x| + \tfrac{3}{8} x^{8/3} + \alpha.$$

b. Since $x^{-4}(x^5+1)^2 = x^6 + 2x + x^{-4}$, it follows that

$$\int \frac{(x^5+1)^2}{x^4}\, dx = \tfrac{1}{7}x^7 + x^2 - \tfrac{1}{3}x^{-3} + \beta.$$

c. Since $\sec(x)(\sec(x) + \tan(x)) = \sec^2(x) + \sec(x)\tan(x)$, it follows that

$$\int \sec(x)(\sec(x) + \tan(x))\, dx = \tan(x) + \sec(x) + \gamma.$$

d. If $0 \leqslant x \leqslant \frac{1}{6}\pi$ then $\sin(x) \leqslant \frac{1}{2}$, and if $\frac{1}{6}\pi \leqslant x \leqslant \frac{1}{2}\pi$ then $\sin(x) \geqslant \frac{1}{2}$. Therefore,

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} \left| \tfrac{1}{2} - \sin(x) \right| dx &= \int_0^{\frac{1}{6}\pi} \left(\tfrac{1}{2} - \sin(x) \right) dx + \int_{\frac{1}{2}\pi}^{\frac{1}{6}\pi} \left(\tfrac{1}{2} - \sin(x) \right) dx \\ &= \left(\tfrac{1}{2}x + \cos(x) \right) \Big|_0^{\frac{1}{6}\pi} + \left(\tfrac{1}{2}x + \cos(x) \right) \Big|_{\frac{1}{2}\pi}^{\frac{1}{6}\pi} \\ &= \sqrt{3} - 1 - \tfrac{1}{12}\pi \end{aligned}$$

17. Barrow's theorem and the chain rule gives

$$\frac{dy}{dx} = \frac{d}{dx} \int_0^{x^2} \frac{t}{1+t^2}\, dt = \frac{x^2}{1+(x^2)^2} \cdot 2x = \frac{2x^3}{1+x^4}.$$

18. If the interval $[0,2]$ is divided into k subintervals of equal length, then the length of each subinterval is $2/k$, the endpoints of the subintervals are $2j/k$ and the values of the integrand at these endpoints are $8j^2/k^2 + 1$, for $j = 0,1,2,\ldots,k$. The Riemann sum obtained by evaluating the integrand at the right endpoints of the subintervals is

$$\begin{aligned} \mathscr{R}_k &= \frac{2}{k} \sum_{j=1}^{k} \left\{ \frac{8j^2}{k^2} + 1 \right\} = \frac{16}{k^3} \sum_{j=1}^{k} j^2 + 2 = \frac{16}{k^3} \cdot \frac{1}{6} k(k+1)(2k+1) + 2 \\ &= \frac{16}{3} \left(1 + \frac{1}{k} \right) \left(1 + \frac{1}{2k} \right) + 2. \end{aligned}$$

Therefore,

$$\int_0^2 (2x^2+1)\, dx = \lim_{k\to\infty} \mathscr{R}_k = \tfrac{16}{3} + 2 = \tfrac{22}{3}.$$

19. The identity function is odd and f is even, so

$$\int_{-2}^{2} x f(x)\, dx = 0 \qquad \text{and} \qquad \int_{-2}^{2} f = 2 \int_0^2 f.$$

Thus, linearity and interval additivity of the definite integral yields

$$\int_{-2}^{2} (x-2) f(x)\, dx = \int_{-2}^{2} x f(x)\, dx - 2 \int_{-2}^{2} f = -4 \int_{-2}^{0} f = -4 \cdot 24 = -96,$$

since

$$\int_{-2}^{0} f = \int_{-2}^{3} f - \int_0^3 f = 240 - 216 = 24.$$