Question 1. — Evaluate the following limits. Use ∞ and $-\infty$ as appropriate.

a. $\displaystyle\lim_{x\to-3}\frac{2x^2+5x-3}{x^3-9x}$ b. $\displaystyle\lim_{x\to3}\frac{\sqrt{1+x}-2}{3-x}$ c. $\displaystyle\lim_{x\to0}\frac{x+1}{\cos(x)-1}$

d. $\displaystyle\lim_{x\to\infty}\sqrt{\frac{1-8x+9x^3}{x^3+x^2+7}}$ e. $\displaystyle\lim_{x\to0}\frac{\sin(6x)\sin(x+\pi/6)}{x}$

Question 2. — Given

$$f(x)=\begin{cases}k(5x-k) & \text{if } x<1,\\ 2k & \text{if } x=1 \text{ and}\\ x^2+3x+2 & \text{if } x>1,\end{cases}$$

a. find all values of k, if any, for which $\lim_{x\to1} f(x)$ is defined;

b. find all values of k, if any, for which f is everywhere continuous.

Question 3. — Give the equations of all asymptotes of the graph of

$$y=\frac{2e^x-5}{e^x-1}.$$

Question 4. — Given $f(x)=\dfrac{1}{4x^2}$.

a. Use the definition of the derivative to compute the derivative of f.

b. Find the point on the graph of f at which the tangent line is parallel to the line defined by $y=1-4x$, and write the equation of this tangent line.

Question 5. — Compute the derivative $\frac{dy}{dx}$ for each of the following. In part c, simplify your answer. Otherwise, do not simplify.

a. $y=\sqrt{x+e^{x\tan x}}$ b. $y=\dfrac{\sec(4x)-2}{\sqrt{x}+1}$

c. $y=(\log x)^{x\log x}$ d. $y=\log\left(\dfrac{(2x+3)^4}{e^{\sin x}\sqrt{x^6-1}}\right)$

Question 6. — Compute and factorize the second derivative of $f(t)=te^{at}$.

Question 7. — Given that $h(x)=f(xg(x))$, $g(2)=-3$, $f'(-6)=4$ and $g'(2)=5$, compute $h'(2)$.

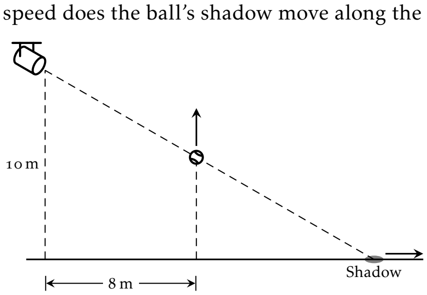
Question 8. — Find all points on the graph of

$$x^2y+xy^2=16$$

where the tangent line is horizontal.

Question 9. — Find the extreme values of $f(x)=\dfrac{4x^2-3}{x^3}$ on $[\frac{1}{2},4]$.

Question 10. — A bright spotlight illuminates a field from a height of 10 m. Eight metres away from the point on the ground below the spotlight, Jack throws a tennis ball vertically into the air and observes how the ball's shadow moves. When the ball reaches a height of 6 m, its speed is 7 m/s. At that instant, at what speed does the ball's shadow move along the ground?

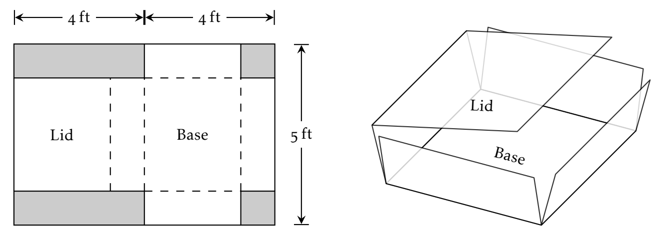


Question 11. — Given $f(x)=\sqrt[3]{2(x-1)(x+2)^2}$,

$$f'(x)=\frac{2x}{\sqrt[3]{4(x-1)^2(x+2)}} \quad\text{and}\quad f''(x)=\frac{-4}{\sqrt[3]{4(x-1)^5(x+2)^4}},$$

sketch the graph of f. Be your solution includes all intercepts, asymptotes, intervals of monotonicity, intervals of concavity, local extrema and points of inflection.

Question 12. — A box with a lid is to be constructed from a 5 ft by 8 ft rectangular piece of cardboard by cutting out squares from two of the corners and strips from the other corners, bending up the sides, and folding the lid across the top to cover the base. Find the dimensions of the box which has the largest possible volume.



Question 13. — Find $f(x)$, given that $f''(x)=x-\cos(x)$, and that the line $y=2x-2$ is tangent to the graph of f at the point where $x=0$.

Question 14. — Express the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i^5}{n^6}-\frac{i^4}{n^5}\right)$$

as the definite integral of a function on $[0,1]$. You need not evaluate the limit.

Question 15. — Evaluate each of the following integrals.

a. $\displaystyle\int\left(\frac{4}{\sqrt[4]{x^7}}+\frac{e^x}{2}+e^{\sqrt{\pi}}\right)dx$ b. $\displaystyle\int_1^4\frac{(2\sqrt{x}-x)^2}{x^3}\,dx$

c. $\displaystyle\int 2(3\sec(x)-5x\cos(x))\sec(x)\,dx$ d. $\displaystyle\int_1^e\frac{d}{dx}\left\{\frac{x\log(x)}{1+x^2}\right\}dx$

Question 16. — Given $A(x)=\displaystyle\int_{1/x}^{2}\frac{t}{t^2+1}\,dt$,

evaluate $A\left(\frac{1}{2}\right)$, and compute and simplify $A'(x)$.

Question 17. — Suppose that f'' is continuous on $\mathbb{R}$, and that $f(0)=2$, $f(1)=5$ and $f(2)=8$.

a. Show that $f'(x)=3$ has at least two real solutions.

b. Show that f'' has a real zero.

Solution to Question 1. — a. Since $2x^2+5x-3=(x+3)(2x-1)$ and $x^3-9x=x(x+3)(x-3)$, it follows that

$$\lim_{x\to-3}\frac{2x^2+5x-3}{x^3-9x}=\lim_{x\to-3}\frac{2x-1}{x(x-3)}=\frac{-7}{(-3)(-6)}=\tfrac{7}{18}.$$

b. Rationalizing the numerator gives

$$\lim_{x\to3}\frac{\sqrt{1+x}-2}{3-x}=\lim_{x\to3}\frac{-1}{\sqrt{1+x}+2}=-\tfrac{1}{4}.$$

c. Since $x+1\to1$ and $\cos(x)-1\to0^-$ as $x\to0$, it follows that

$$\lim_{x\to0}\frac{x+1}{\cos(x)-1}=-\infty.$$

d. Inspecting the dominant terms gives

$$\lim_{x\to\infty}\sqrt{\frac{1-8x+9x^3}{x^3+x^2+7}}=\sqrt{9}=3.$$

e. Multiplying and dividing by 6 yields

$$\begin{aligned}\lim_{x\to0}\frac{\sin(6x)\sin(x+\pi/6)}{x}&=\lim_{x\to0}\left\{\frac{\sin(6x)}{6x}\cdot6\sin(x+\pi/6)\right\}\\&=1\cdot6\sin(\pi/6)=6\cdot\tfrac12=3.\end{aligned}$$

Solution to Question 2. — Since

$$\lim_{x\to1^-}f(x)=k(5-k)\qquad\text{and}\qquad\lim_{x\to1^+}f(x)=6,$$

it follows that $\lim_{x\to1}f(x)$ is defined if, and only if, $k(5-k)=6$, or $0=k^2-5k+6=(k-2)(k-3)$, *i.e.*, $k=2,3$. Next, the function f is everywhere continuous if, in addition $f(1)=6$, or $2k=6$, *i.e.*, $k=3$.

Solution to Question 3. — Since

$$\lim_{x\to0^\pm}\frac{2e^x-5}{e^x-1}=\mp\infty,\quad\lim_{x\to-\infty}\frac{2e^x-5}{e^x-1}=5\quad\text{and}\quad\lim_{x\to\infty}\frac{2e^x-5}{e^x-1}=2,$$

it follows that the horizontal asymptotes of the graph are defined by $y=5$ and $y=2$, and the vertical asymptote of the graph is defined by $x=0$.

Solution to Question 4. — If $y=\frac14x^{-2}$ and $y'=\frac14x'^{-2}$, then

$$y'-y=\frac{x^2-x'^2}{4x'^2x^2}y'-y=\frac{(x-x')(x+x')}{4x'^2x^2}$$

so

$$f'(x)=\frac{dy}{dx}=\lim_{x'\to x}\frac{y'-y}{x'-x}=\lim_{x'\to x}\frac{-(x+x')}{4x'^2x^2}=\frac{-2x}{4x^4}=-\frac{1}{2x^3}.$$

The tangent line to the graph of f is parallel to the line defined by $y=1-4x$ where $f'(x)=-4$, *i.e.*, $x^3=\frac18$ or $x=\frac12$, in which case $y=\frac14\cdot2^2=1$. Thus, the tangent line to the graph of f which is parallel to the line defined by $y=1-4x$, is defined by $4x+y=3$.

Solution to Question 5. — a. If $y=\sqrt{x+e^{x\tan x}}$, then

$$\frac{dy}{dx}=\frac{1+e^{x\tan x}(\tan(x)+x\sec^2(x))}{2\sqrt{x+e^{x\tan x}}}.$$

b. If $y=\dfrac{\sec(4x)-2}{\sqrt{x}+1}$, then $\dfrac{dy}{dx}=\dfrac{4\sec(4x)\tan(4x)}{1+\sqrt{x}}-\dfrac{\sec(4x)-2}{2(1+\sqrt{x})^2\sqrt{x}}$.

c. If $y=(\log x)^{x\log x}$, then

$$\frac{dy}{dx}=y\frac{d}{dx}\big(\log y\big)=(\log x)^{x\log x}\big(\log(x)\log(\log(x))+\log(\log(x))+1\big).$$

d. If $y=\log\left(\dfrac{(2x+3)^4}{e^{\sin x}\sqrt{x^6-1}}\right)$, then $\dfrac{dy}{dx}=\dfrac{8}{2x+3}-\cos(x)-\dfrac{3x^5}{x^6-1}$.

Solution to Question 6. — If $f(t)=te^{at}$, then $f'(t)=e^{at}(1+at)$ and

$$f''(t)=e^{at}(a+a+a^2t)=ae^{at}(2+at)=a^2e^{at}(t+2/a).$$

Solution to Question 7. — The chain and product rules give

$$h'(2)=f'(2g(2))\big(g(2)+2\cdot g'(2)\big)=f'(-6)(-3+10)=4\cdot7=28.$$

Solution to Question 8. — On the curve defined by $x^2y+xy^2=16$

$$\frac{dy}{dx}=-\frac{2xy+y^2}{x^2+2xy}=-\frac{y(2x+y)}{x(x+2y)},$$

which is equal to zero if, and only if, $y(2x+y)=0$ and $x(x+2y)\neq0$. As there are no points on the curve where $y=0$, it follows that $y=-2x$, and so $16=-2x^3+4x^3=2x^3$, *i.e.*, $x^3=8$, or $x=2$. Thus, the tangent line to the given curve is horizontal at the point $(2,-4)$ and nowhere else.

Solution to Question 9. — If $f(x)=\dfrac{4x^2-3}{x^3}=\dfrac4x-\dfrac{3}{x^3}$ then

$$f'(x)=-\frac{4}{x^2}+\frac{9}{x^4}=\frac{9-4x^2}{x^4},$$

so the critical numbers of f are $\pm\frac32$, of which only $\frac32$ lies in the given interval. Comparing

$$f\big(\tfrac12\big)=8-24=-16,\quad f\big(\tfrac32\big)=\tfrac83-\tfrac89=\tfrac{16}{9}\quad\text{and}\quad f(4)=1-\tfrac{3}{64}=\tfrac{61}{64},$$

reveals that the largest value of f on the interval $[\frac12,4]$ is $\frac{16}{9}$ and the smallest value is -16.

Solution to Question 10. — If y denotes the height of the ball and x denotes the distance between the shadow and the point on the ground directly beneath the ball, then $y=10x/(x+8)=10-80/(x+8)$, so

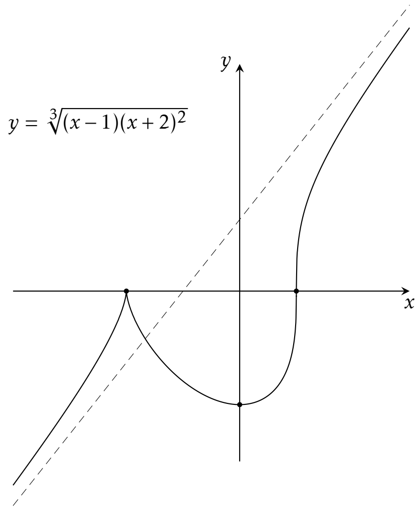
$$\frac{dy}{dt}=\frac{80}{(x+8)^2}\frac{dx}{dt}.$$

When the ball is 6 metres high, $6(x+8)=10x$, or $x=12$, so $7=\frac{80}{400}\frac{dx}{dt}$, or $\frac{dx}{dt}=35$. Therefore, at the given instant the ball's shadow is moving along the ground at a rate of 35 metres per second.

Solution to Question 11. — The domain of f is $\mathbb{R}$ on which f is continuous, so the graph has no vertical asymptotes. The intercepts are $(0,2)$, $(-2,0)$ and $(1,0)$. If $a=f(x)$ and $b=\sqrt[3]{2}x$ (the dominant term of $f(x)$), then $a^3-b^3=2(x-1)(x+2)^2-2x^3=2(3x^2-4)$, and the dominant term of a^2+ab+b^2 is $3\sqrt[3]{4}x^2$, so

$$\begin{aligned}\lim_{x\to\pm\infty}\big\{f(x)-\sqrt[3]{2}x\big\}&=\lim_{x\to\pm\infty}\frac{a^3-b^3}{a^2+ab+b^2}=\lim_{x\to\pm\infty}\frac{2(3x^2-4)}{a^2+ab+b^2}\\&=\frac{2\cdot3}{3\sqrt[3]{4}}=\sqrt[3]{2}.\end{aligned}$$

as is seen by inspecting the dominant terms, so the oblique asymptote of the graph is defined by $y=\sqrt[3]{2}(x+1)$. The derivative $f'(x)$ is positive if $x<-2$ or $x>0$ and $x\neq1$, and is negative if $-2<x<0$, so f is increasing on the intervals $(-\infty,-2]$ and $[0,\infty)$, and decreasing on the interval $[-2,0]$, with a local maximum at $(-2,0)$ and a local minimum at $(0,-2)$. The second derivative $f''(x)$ is positive if $-2<x<1$ and positive if $x<-2$ or $x>1$, so the graph of f is concave up on the interval $[-2,1]$ and concave down on the intervals $(-\infty,-2]$ and $[1,\infty)$, with a point of inflection at $(1,0)$. Below is a sketch of the graph of f, with the asymptote drawn as a dashed line, and the points of interest emphasized.



Solution to Question 12. — The volume of the box is given by $V = (5 - 2x)(4 - x)x = 2x^3 - 13x^2 + 20x$, where x is the side of the squares cut from two of the corners. Now

$$\frac{dV}{dx} = 6x^2 - 26x + 20 = 2(3x^2 - 13x + 10) = 2(3x - 10)(x - 1),$$

which is positive if $0 < x < 1$ and negative if $1 < x < \frac{5}{2}$ (beyond which there is no base), so the volume of the box is maximized if $x = 1\,\text{ft}$, in which case the base of the box is a $3\,\text{ft}$ by $3\,\text{ft}$ square.

Solution to Question 13. — The tangency condition implies that $f'(0) = 2$ and $f(0) = -2$, so that $f'(x) = \frac{1}{2}x^2 - \sin(x) + 2$ and $f(x) = \frac{1}{6}x^3 + \cos(x) + 2x - 3$.

Solution to Question 14. — Revising the expression in the limit gives

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(2\left(\frac{i}{n}\right)^5 - \left(\frac{i}{n}\right)^4\right) = \int_0^1 (2x^5 - x^4)\,dx.$$

Solution to Question 15. — a. Integrating by inspection gives

$$\int\left(\frac{4}{\sqrt[4]{x^7}} + \frac{e^x}{2} + e^{\sqrt{\pi}}\right)dx = -\tfrac{16}{3}x^{-3/4} + \tfrac{1}{2}e^x + e^{\sqrt{\pi}}x.$$

b. Expanding and dividing gives

$$\begin{aligned}\int_1^4 \frac{(2\sqrt{x} - x)^2}{x^3}\,dx &= \int_1^4 \left\{4x^{-2} - 4x^{-3/2} + x^{-1}\right\}dx \\ &= \left(-\frac{4}{x} + \frac{8}{\sqrt{x}} + \log(x)\right)\Big|_1^4 \\ &= \log(4) - 1 = \log(4/e).\end{aligned}$$

c. Expanding and integrating by inspection gives

$$\begin{aligned}\int 2(3\sec(x) - 5x\cos(x))\sec(x)\,dx &= \int\left(6\sec^2(x) - 10x\right)dx \\ &= 6\tan(x) - 5x^2.\end{aligned}$$

d. Integrating by inspection gives

$$\int_1^e \frac{d}{dx}\left\{\frac{x\log(x)}{1 + x^2}\right\}dx = \frac{x\log(x)}{1 + x^2}\Big|_1^e = \frac{e}{1 + e^2}.$$

Solution to Question 16. — Basic properties of the definite integral and Barrow's formula give

$$A\left(\tfrac{1}{2}\right) = \int_2^2 \frac{t}{t^2 + 1}\,dt = 0$$

and

$$A'(x) = -\frac{d}{dx}\int_2^{1/x} \frac{t}{t^2 + 1}\,dt = \frac{-1/x}{(1/x)^2 + 1}\cdot\frac{-1}{x^2} = \frac{1}{x(1 + x^2)}.$$

Solution to Question 17. — a. Since f'' is everywhere continuous, the mean value theorem may be applied to f on any closed interval of positive length. Hence, there are real numbers ξ, ξ' such that $0 < \xi < 1$, $1 < \xi' < 2$,

$$f'(\xi) = \frac{f(1) - f(0)}{1 - 0} = 3 \qquad\text{and}\qquad f'(\xi') = \frac{f(2) - f(1)}{2 - 1} = 3,$$

which shows that $f'(x) = 3$ has (at least) two real solutions.

b. Since f'' is everywhere continuous, the mean value theorem may be applied to f' on the closed interval $[\xi, \xi']$, so there is a real number η such that $\xi < \eta < \xi'$ and

$$f''(\eta) = \frac{f'(\xi) - f'(\xi')}{\xi' - \xi} = \frac{3 - 3}{\xi' - \xi} = 0.$$

Therefore, f'' has at least one real zero.