Question 1. — Evaluate the following limits.

a. $\displaystyle\lim_{x\to 2^-} \frac{2x^3-4x^2}{3x^2-8x+4}$ b. $\displaystyle\lim_{x\to 0} \frac{\sin^2(3x)}{5x\sin(2x)}$

c. $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{4x^6+3x^5}}{2x^3+\sqrt{9x^6+7x^5}}$ d. $\displaystyle\lim_{x\to 2^+} \frac{\sqrt{x-2}-x+2}{6-3x}$

Question 2. — Let

$$f(x) = \begin{cases} x^2-k-3 & \text{if } x<-1, \\ k+4 & \text{if } x=-1 \text{ and} \\ k^2+4x-4 & \text{if } x>-1. \end{cases}$$

Find all values of k such that:

a. $\displaystyle\lim_{x\to -1} f(x)$ is defined; b. f is continuous on $\mathbb{R}$.

Question 3. — Use the limit definition of the derivative to find $f'(x)$, where

$$f(x) = \frac{1}{3-2x}.$$

Question 4. — Find $\frac{dy}{dx}$ for each of the following. Do not simplify your answers.

a. $y = 16\sqrt[4]{x} + e^x - x^e + \dfrac{\pi}{x}$ b. $y = \dfrac{(8-5x^2)^4}{\tan(7x)-9}$

c. $y = e^{\sqrt{2x^3}}$ d. $y = (\sin x)^{4\ln x}$

Question 5. — Write an equation of the line tangent to the curve defined by

$$x^2y + \sin(y) + \frac{4}{\pi}y = 3e^x$$

at the point $(0, \frac{1}{2}\pi)$.

Question 6. — Let ϑ be the radian measure of an acute angle in a right-angled triangle and let x and y be, respectively, the lengths of the sides adjacent and opposite to ϑ. Suppose also that x and y vary with time. At a certain instant, $x = 4\,\text{cm}$ and is increasing at $8\,\text{cm/s}$, while $y = 3\,\text{cm}$ and is decreasing at $2\,\text{cm/s}$. At what rate is ϑ changing at that instant?

Question 7. — A box with a square base and open top needs to be made. The material for the base of the box costs \$10 per square metre, while the material for the sides costs \$5 per square metre. Using only \$120, what are the dimensions of such a box with largest volume?

Question 8. — Find the absolute extrema of $f(x) = \dfrac{x}{2} + \dfrac{2}{x^2}$ on $[1,4]$.

Question 9. — The linear position of a particle is given by $s = t^3 - 3t^2$, where s is measured in metres and $t \geqslant 0$ is measured in seconds.

a. Find the velocity function of the particle.
b. At what times is the particle at rest?
c. When is the particle moving in the positive direction?

Question 10. — Sketch the graph of $f(x) = \dfrac{x+2}{\sqrt{x^2+2}}$, given that

$$f'(x) = \frac{2(1-x)}{(x^2+2)^{3/2}} \qquad \text{and} \qquad f''(x) = \frac{2(x-2)(2x+1)}{(x^2+2)^{5/2}}.$$

Make sure that your solution includes all intercepts, asymptotes, intervals of monotonicity, intervals of concavity, local extrema and points of inflection.

Question 11. — Evaluate each of the following integrals.

a. $\displaystyle\int \left(\frac{2}{x} - \sqrt[3]{x^5} + 7e^x\right) dx$ b. $\displaystyle\int \frac{(5x-3)^2}{x}\, dx$

c. $\displaystyle\int \frac{1-\sin(\vartheta)}{\cos^2(\vartheta)}\, d\vartheta$ d. $\displaystyle\int_2^3 \frac{x^2+8x+15}{x+3}\, dx$

Question 12. — Given $f(x) = \displaystyle\int_6^{1/x} \frac{t}{\sqrt{1+t}}\, dt$, find $f(1/6)$ and $f'(x)$.

Question 13. — Express

$$\int_0^5 \sin(x^2)\, dx$$

as a limit of Riemann sums. Do not evaluate the limit.

Question 14. — Decide whether or not the equality below is correct. Justify your answer.

$$\int \log(x)\, dx = x\log(x) - x.$$

Solution to Question 1. — a. Factorizing gives

$$\lim_{x\to 2^-}\frac{2x^2(x-2)}{(x-2)(3x-2)}=\frac{8}{4}=2.$$

b. Revising the expression gives

$$\lim_{x\to 0}\left\{\left(\frac{\sin 3x}{3x}\right)^2\frac{2x}{\sin(2x)}\cdot\frac{3^2}{2\cdot 5}\right\}=\tfrac{9}{10}.$$

c. Inspecting dominant terms gives (recall that $\sqrt{x^6}=-x^3$ if $x<0$)

$$\lim_{x\to-\infty}\frac{\sqrt{4x^6+3x^5}}{2x^3+\sqrt{9x^6+7x^5}}=\frac{-\sqrt{4}}{2-\sqrt{9}}=2.$$

d. Factorizing gives

$$\lim_{x\to 2^+}\frac{\sqrt{x-2}\,(1-\sqrt{x-2}\,)}{-3(x-2)}=\lim_{x\to 2^+}\frac{1-\sqrt{x-2}}{-3\sqrt{x-2}}=-\infty,$$

since $1-\sqrt{x-2}\to 1$ and $-3\sqrt{x-2}\to 0^-$ as $x\to 2^+$.

Solution to Question 2. — Since

$$\lim_{x\to-1^-}f(x)=-k-2,\quad f(-1)=k+4\quad\text{and}\quad\lim_{x\to-1^+}f(x)=k^2-8,$$

it follows that $\lim_{x\to-1}f(x)$ is defined if, and only if,

$$k^2-8=-k-2,\qquad\text{or}\qquad 0=k^2+k-6=(k+3)(k-2),$$

i.e., $k=-3,2$.

The function f is everywhere continuous if, in addition, $k+4=-k-2$, or $k=-3$.

Solution to Question 3. — If $y=1/(3-2x)$ then $y'=1/(3-2x')$ and

$$y'-y=\frac{3-2x-(3-2x')}{(3-2x')(3-2x)}=\frac{2(x'-x)}{(3-2x')(3-2x)},$$

so

$$f'(x)=\lim_{x'\to x}\frac{y'-y}{x'-x}=\lim_{x'\to x}\frac{2}{(3-2x')(3-2x)}=\frac{2}{(3-2x)^2}.$$

Solution to Question 4. — a. If $y=16\sqrt[4]{x}+e^x-x^e+\dfrac{\pi}{x}$, then

$$\frac{dy}{dx}=4x^{-3/4}+e^x-ex^{e-1}-\pi/x^2.$$

b. If $y=\dfrac{(8-5x^2)^4}{\tan(7x)-9}$, then

$$\frac{dy}{dx}=\frac{-40x(8-5x)^3}{\tan(7x)-9}-\frac{7(8-5x^2)^4\sec^2(7x)}{(\tan(7x)-9)^2}.$$

c. If $y=e^{\sqrt{2x^3}}$, then $\dfrac{dy}{dx}=\tfrac{3}{2}\sqrt{2x}\,e^{\sqrt{2x^3}}$.

d. If $y=(\sin x)^{4\ln x}$, then

$$\frac{dy}{dx}=y\frac{d}{dx}\big(\log y\big)=4(\sin x)^{4\ln x}\left\{\frac{\log(\sin x)}{x}+\cot(x)\log(x)\right\}.$$

Solution to Question 5. — The slope of the tangent line to the curve defined by $x^2y+\sin(y)+\frac{4}{\pi}y-3e^x=0$ is

$$\left.\frac{dy}{dx}\right|_{\substack{x=0\\ y=\frac{1}{2}\pi}}=-\left.\frac{2xy-3e^x}{x^2+\cos(y)+4/\pi}\right|_{\substack{x=0\\ y=\frac{1}{2}\pi}}=\tfrac{3}{4}\pi,$$

so the tangent line is defined by $y=\frac{3}{4}x+\frac{1}{2}\pi$.

Solution to Question 6. — Since $\tan(\vartheta)=y/x$, differentiating with respect to time gives

$$\big(1+\tan^2(\vartheta)\big)\frac{d\vartheta}{dt}=\frac{1}{x}\frac{dy}{dt}-\frac{y}{x^2}\frac{dx}{dt}.$$

If $x=4$ and $y=3$ then $1+\tan^2(\vartheta)=1+\left(\frac{3}{4}\right)^2=\frac{25}{16}$, so at the given instant

$$\frac{d\vartheta}{dt}=\tfrac{16}{25}\left(\tfrac{1}{4}\cdot(-2)-\tfrac{3}{16}\cdot 8\right)=-\tfrac{32}{25}.$$

Therefore, at the instant in question, ϑ is decreasing at a rate of $\frac{32}{25}$ radians per second.

Solution to Question 7. — If x denotes the side of the base of the box and y its height, then $120=10x^2+20xy$, so $y=6/x-x/2$, and the volume of the box is $V=x^2y=6x-\frac{1}{2}x^3$. Then

$$\frac{dV}{dx}=6-\tfrac{3}{2}x^2=\tfrac{3}{2}(4-x^2),$$

which is positive if $0<x<2$ and negative if $2<x<2\sqrt{3}$ (beyond which y is negative), so V is maximized if $x=2$ and $y=6/2-2/2=2$. Hence, the largest such box is a cube with side 2 metres.

Solution to Question 8. — The derivative of f is

$$f'(x)=\frac{1}{2}-\frac{4}{x^3}=\frac{8-x^2}{2x^3},$$

so the only critical number of f is 2. Comparing

$$f(1)=\tfrac{1}{2}+2=\tfrac{5}{2},\qquad f(2)=1+\tfrac{1}{2}=\tfrac{3}{2}\qquad\text{and}\qquad f(4)=2+\tfrac{1}{8}=\tfrac{17}{8},$$

reveals that the largest value of f on $[1,4]$ is $\frac{5}{2}$ and the smallest value is $\frac{3}{2}$.

Solution to Question 9. — The velocity of the particle is

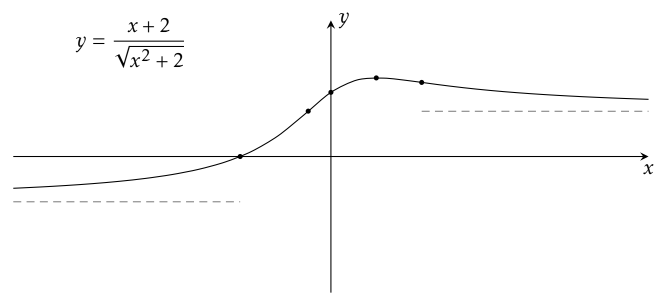
$$\frac{ds}{dt}=3t^2-6t=3t(t-2),$$

so the particle is a rest when $t=0,2$ and the particle moves in the positive direction when $t>2$.

Solution to Question 10. — The domain of f is $\mathbb{R}$, the intercepts are $(0,\sqrt{2})$ and $(-2,0)$, the graph has no vertical asymptotes, and the horizontal asymptotes are defined by $y=\pm 1$, since

$$\lim_{x\to\pm\infty}\frac{x+2}{\sqrt{x^2+2}}=\pm 1,$$

as is seen by inspecting the dominant terms. The first derivative is positive if $x<1$ and negative if $x>1$, so f is increasing on the interval $(-\infty,1\,]$, decreasing on the interval $[\,1,\infty)$, with a local (and global, as it turns out) maximum at $(1,\sqrt{3})$. The second derivative is positive if $x<-\frac{1}{2}$ or $x>2$ and negative if $-\frac{1}{2}<x<2$, so the graph is concave up on the intervals $(-\infty,-\frac{1}{2}\,]$ and $[\,2,\infty)$, concave down on the interval $[-\frac{1}{2},2\,]$ with points of inflection $(-\frac{1}{2},1)$ and $(2,\frac{2}{3}\sqrt{6})$. Below is a sketch of the graph of f, with the horizontal asymptotes drawn as dashed lines and the points of interest emphasized.



Solution to Question 11. — a. Integrating by inspection gives

$$\int\left(\frac{2}{x}-\sqrt[3]{x^5}+7e^x\right)dx=2\log|x|-\tfrac{3}{8}x^{8/3}+7e^x.$$

b. Expanding, dividing and integrating by inspection gives

$$\int\frac{(5x-3)^2}{x}\,dx=\int\left(25x-30+\frac{9}{x}\right)dx=\tfrac{25}{2}x^2-30x+9\log|x|.$$

c. Dividing and integrating by inspection gives

$$\int\frac{1-\sin(\vartheta)}{\cos^2(\vartheta)}\,d\vartheta=\int\left(\frac{1}{\cos^2(\vartheta)}-\frac{\sin(\vartheta)}{\cos^2(\vartheta)}\right)d\vartheta=\tan(\vartheta)-\frac{1}{\cos(\vartheta)}.$$

d. Since $x^2+8x+15=(x+3)(x+5)$, it follows that

$$\int_2^3\frac{x^2+8x+15}{x+3}\,dx=\int_2^3(x+5)\,dx=\tfrac{1}{2}(x+5)^2\Big|_2^3=\tfrac{1}{2}(64-49)=\tfrac{15}{2}.$$

Solution to Question 12. — Interval additivity and Barrow's theorem give

$$f(1/6) = \int_6^6 \frac{t}{\sqrt{1+t}}\,dt = 0$$

and

$$f'(x) = \frac{1/x}{\sqrt{1+1/x}} \cdot \frac{-1}{x^2} = \frac{-1}{x^3\sqrt{1+1/x}}.$$

Solution to Question 13. — If the interval $[0,5]$ is divided into k subintervals of equal length, then the length of each subinterval is $\frac{5}{k}$ and the endpoints of the subintervals are $\frac{5}{k}j$, for $j = 0, 1, 2, 3, \ldots, k$. If right endpoints are marked for evaluation, then

$$\int_0^5 \sin(x^2)\,dx = \lim_{k\to\infty} \frac{5}{k} \sum_{j=1}^{k} \sin\left(\frac{25j^2}{k^2}\right).$$

Solution to Question 14. — Since

$$\frac{d}{dx}\Big(x\log(x) - x\Big) = \log(x) + x \cdot 1/x - 1 = \log(x),$$

it follows that the equation $\int \log(x)\,dx = x\log(x) - x$ is correct.