Question 1. — Evaluate the following limits. Use $-\infty$, ∞ or "does not exist" where appropriate.

a. $\displaystyle\lim_{x\to 2}\frac{\frac{3}{x+1}-\frac{4}{x^2}}{x-2}$ b. $\displaystyle\lim_{x\to e^+}\frac{\log(x)-3}{1-\log(x)}$ c. $\displaystyle\lim_{x\to 4^+}\frac{|x^2-16|-|8-2x|}{4-x}$

d. $\displaystyle\lim_{x\to 0}\frac{\sin(x)+\sin(3x)}{2x}$ e. $\displaystyle\lim_{x\to 0^+}\left\{x^{2/3}\cos\big(2/(3x)\big)-2\right\}$

Question 2. — Give the equations of all asymptotes of the graph of

$$y=\frac{3x+5+\sqrt{4x^2+1}}{2x-\sqrt{x^2+27}}.$$

Question 3. — Let

$$f(x)=\begin{cases}ax+5 & \text{if } x<5,\\ 2^{a+3}-6 & \text{if } x=5 \text{ and}\\ \dfrac{x^2-25}{ax-5a} & \text{if } x>5.\end{cases}$$

a. Find all values of a, if any, for which $\lim_{x\to 5} f(x)$ is defined.

b. Find all values of a, if any, for which f is continuous at 5.

Question 4. — Given $y=\sqrt{10-x^2}$.

a. Find $\frac{dy}{dx}$ using the limit definition of the derivative.
b. Check your answer from part a using the rules of differentiation.
c. Write an equation of the tangent line at the point where $x=1$.

Question 5. — Compute $\frac{dy}{dx}$. Do not simplify your answers.

a. $y=5^x+x^5+\dfrac{1}{\sqrt{x^5}}+\dfrac{1}{\sqrt{5}}$ b. $y=e^{5x^2}\cos(\log_2(x))$

c. $y=\dfrac{(x^3+1)^4}{\tan^6(2x)+10}$ d. $y=\big(x\sec(x)\big)^{x^2}$ e. $x^2-\cos(xe^y)=3xy$

Question 6. — Use logarithmic differentiation to compute $\frac{dy}{dx}$, where

$$y=\frac{3\csc^4(x)}{x^5\sqrt[6]{\log(x)}}.$$

Question 7. — Find the 96^{th} derivative of the function defined by

$$f(x)=\frac{x}{7-x}.$$

Question 8. — An elastic cylindrical tube is being stretched along its axis by pulling on its ends. The object retains a (circular) cylindrical shape as it is stretched, and the volume remains constant at $40\pi\,\text{cm}^3$. At a given instant, the length of the cylinder is observed to be 10 cm but increasing at the rate of 5 cm/min. What is the rate of change of the radius at that instant?

Question 9. — Sketch the graph of f, given

$$f(x)=\frac{2(x-3)^3}{27(x-1)},\quad f'(x)=\frac{4x(x-3)^2}{27(x-1)^2}\quad\text{and}\quad f''(x)=\frac{4(x-3)(x^2+3)}{27(x-1)^3}.$$

Make sure that your solution includes all intercepts, asymptotes, intervals of monotonicity, intervals of concavity, local extrema and points of inflection.

Question 10. — You are designing a new rectangular sign for the next climate strike. You need a rectangular region of $24\,\text{m}^2$ for your witty slogan, and you need to leave a blank space of $\frac{1}{2}$ m wide on each side and $\frac{1}{3}$ m on the top and bottom of the region for your slogan. What should be the dimensions (width and height) of the region for your climate slogan in order to minimize the total are of cardboard needed to make the sign?

Question 11. — Find the extreme values of $f(x)=(x-32)\sqrt[3]{x}$ for $-8\leqslant x\leqslant 1$.

Question 12. — A particle moves with acceleration $a=e^t+4\sin(t)-2\cos(t)$. Find the position of the particle as a function of time if its initial position is 0 and its initial velocity is 2.

Question 13. — Compute each integral below.

a. $\displaystyle\int\frac{(\sqrt{x}-3)(2x-1)}{\sqrt{x^3}}\,dx$ b. $\displaystyle\int_2^4\left\{\left(\frac{2}{y}\right)^2+e^2\right\}dy$ c. $\displaystyle\int\frac{5\sin(x)-e^x\cos^2(x)}{\cos^2(x)}\,dx$

Question 14. — Consider the function defined by

$$f(x)=\begin{cases}|x|-2 & \text{if } -3\leqslant x\leqslant 3, \text{ and}\\ 1 & \text{elsewhere.}\end{cases}$$

Sketch the graph of f and use it to calculate $\displaystyle\int_{-5}^{5} f$.

Question 15. — Evaluate the integral

$$\int_0^5(4-x^2)\,dx$$

by expressing it as a limit of Riemann sums.

Question 16. — Compute $F'(x)$, where $F(x)=\displaystyle\int_{e^{2x}}^{\pi} y\tan(y)\,dy$.

Question 17. — Use the Intermediate Value Theorem to show that the equation $x^4-x^3-x=4$ has at least one positive and one negative root.

Solution to Question 1. — a. Since $3x^2-4(x+1)=(x-2)(3x+2)$, it follows that

$$\lim_{x\to 2}\frac{\dfrac{3}{x+1}-\dfrac{4}{x^2}}{x-2}=\lim_{x\to 2}\frac{3x+2}{x^2(x+1)}=\frac{8}{4\cdot 3}=\tfrac{2}{3}.$$

b. As $x\to e$ and $x>e$, $\log(x)-3\to -2$ and $1-\log(x)\to 0$ and $1-\log(x)<0$, so

$$\lim_{x\to e^+}\frac{\log(x)-3}{1-\log(x)}=\infty.$$

c. If $x>4$ then $|x^2-16|-|8-2x|=x^2-16-(2x-8)=(x-4)(x+2)$, so that

$$\lim_{x\to 4^+}\frac{|x^2-16|-|8-2x|}{4-x}=-\lim_{x\to 4^+}(x+2)=-6.$$

d. Since $(\sin\vartheta)/\vartheta\to 1$ as $\vartheta\to 0$, it follows that

$$\lim_{x\to 0}\frac{\sin(x)+\sin(3x)}{2x}=\lim_{x\to 0}\left\{\frac{1}{2}\cdot\frac{\sin(x)}{x}+\frac{3}{2}\cdot\frac{\sin(3x)}{3x}\right\}=\tfrac{1}{2}+\tfrac{3}{2}=2.$$

e. Since $\left|x^{2/3}\cos\big(2/(3x)\big)\right|\leqslant x^{2/3}$, it is plain (from the definition of limit) that

$$\lim_{x\to 0^+}\left\{x^{2/3}\cos\big(2/(3x)\big)-2\right\}=0-2=-2.$$

Solution to Question 2. — Inspecting dominant terms gives

$$\lim_{x\to-\infty}\frac{3x+5+\sqrt{4x^2+1}}{2x-\sqrt{x^2+27}}=\frac{3-2}{2+1}=\tfrac{1}{3}$$

and

$$\lim_{x\to\infty}\frac{3x+5+\sqrt{4x^2+1}}{2x-\sqrt{x^2+27}}=\frac{3+2}{2-1}=5,$$

so the horizontal asymptotes of the graph are defined by $y=\frac{1}{3}$ and $y=5$. Next, $2x-\sqrt{x^2+27}=0$ if, and only if, $x>0$ and $4x^2=x^2+27$, or $x^2=9$; *i.e.*, $x=3$. Now $2x>\sqrt{x^2+27}$ if $x>3$, $2x<\sqrt{x^2+27}$ if $0<x<3$ and $3\cdot 3+5+\sqrt{4\cdot 9+1}=14+\sqrt{37}>0$, so

$$\lim_{x\to 3^-}\frac{3x+5+\sqrt{4x^2+1}}{2x-\sqrt{x^2+27}}=-\infty\quad\text{and}\quad\lim_{x\to 3^-}\frac{3x+5+\sqrt{4x^2+1}}{2x-\sqrt{x^2+27}}=\infty.$$

Thus, the vertical asymptote of the graph is defined by $x=3$.

Solution to Question 3. — First of all,

$$\lim_{x\to 5^-}f(x)=5(a+1)\qquad\text{and}\qquad\lim_{x\to 5^+}f(x)=\lim_{x\to 5^+}\frac{x+5}{a}=\frac{10}{a},$$

so the limit of $f(x)$ as $x\to 5$ is defined if, and only if, $a\neq 0$ and $a^2+a=2$; *i.e.*, $a=1$ or $a=-2$. If $a=1$ then $\lim_{x\to 5}f(x)=10$ and $f(5)=2^4-6=10$, and if $a=-2$ then $\lim_{x\to 5}f(x)=-5$ and $f(5)=2-6=-4$. Therefore, f is continuous at 5 if, and only if, $a=1$.

Solution to Question 4. — If $y=\sqrt{10-x^2}$ and $y'=\sqrt{10-x'^2}$, then $y'^2-y^2=x^2-x'^2$, and thus $y'-y=-(x'-x)(x'+x)/(y'+y)$. Therefore, by the definition of the derivative,

$$\frac{dy}{dx}=\lim_{x'\to x}\frac{y'-y}{x'-x}=-\lim_{x'\to x}\frac{x'+x}{y'+y}=-\frac{x}{y}=\frac{-x}{\sqrt{10-x^2}},$$

which is defined provided $|x|<\sqrt{10}$. The same result is obtained using the power rule and the chain rule:

$$\frac{d}{dx}\left\{\sqrt{10-x^2}\right\}=\tfrac{1}{2}(10-x^2)^{-1/2}(-2x)=\frac{-x}{\sqrt{10-x^2}}.$$

Finally, if $x=1$ then $y=\sqrt{9}=3$ and $\frac{dy}{dx}=-x/y=-\frac{1}{3}$; so the tangent line is defined by $x+3y=10$.

Solution to Question 5. — a. If $y=5^x+x^5+\dfrac{1}{\sqrt{x^5}}+\dfrac{1}{\sqrt{5}}$, then

$$\frac{dy}{dx}=5^x\log(5)+5x^4-\tfrac{5}{2}x^{-7/2}.$$

b. If $y=e^{5x^2}\cos(\log_2(x))$, then

$$\frac{dy}{dx}=e^{5x^2}\left\{10x\cos(\log_2(x))-\sin(\log_2(x))/(x\log(2))\right\}.$$

c. If $y=\dfrac{(x^3+1)^4}{\tan^6(2x)+10}$, then

$$\frac{dy}{dx}=\frac{12x^2(x^3+1)^3}{\tan^6(2x)+10}-\frac{12(x^3+1)^4\tan^5(2x)\sec^2(2x)}{(\tan^6(2x)+10)^2}.$$

d. If $y=\big(x\sec(x)\big)^{x^2}$, then

$$\frac{dy}{dx}=\big(x\sec(x)\big)^{x^2}\left\{2x\log(x\sec(x))+x+x^2\tan(x)\right\}.$$

e. If $x^2-\cos(xe^y)=3xy$, then

$$\frac{dy}{dx}=-\frac{2x+e^y\sin(xe^y)-3y}{xe^y\sin(xe^y)-3x}.$$

Solution to Question 6. —

$$\frac{dy}{dx}=y\frac{d}{dx}\left\{\log(y)\right\}=-\frac{3\csc^4(x)}{x^5\sqrt[6]{\log(x)}}\left\{4\cot(x)+\frac{5}{x}+\frac{1}{6x\log(x)}\right\}.$$

Solution to Question 7. — If $y=\dfrac{x}{7-x}=-1+\dfrac{7}{7-x}$, then

$$\frac{d^ny}{dx^n}=\frac{7\cdot 1\cdot 2\cdot 3\cdots n}{(7-x)^{n+1}}=\frac{7\cdot n!}{(7-x)^{n+1}},$$

so in particular $\dfrac{d^{96}y}{dx^{96}}=\dfrac{7\cdot 96!}{(7-x)^{97}}$.

Solution to Question 8. — If x denotes the radius and y denotes the length (each in centimetres) of the tube then $x^2y=40$, $x=2$ and $\frac{dy}{dx}=5$ when $y=10$, and

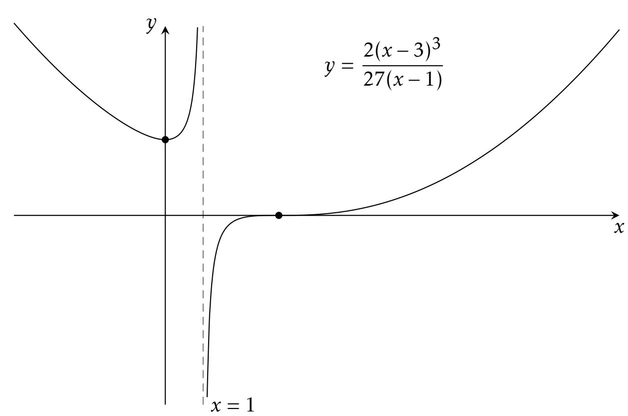
$$2xy\frac{dx}{dt}+x^2\frac{dy}{dt}=0;$$

therefore,

$$\left.\frac{dx}{dt}\right|_{y=10}=-\frac{x}{2y}\left.\frac{dy}{dt}\right|_{y=10}=-\frac{2}{2\cdot 10}\cdot 5=-\tfrac{1}{2}.$$

The radius of the tube is decreasing by $\frac{1}{2}$ cm/min at the instant in question.

Solution to Question 9. — The domain of f is the set of all real numbers except 1, the intercepts of the graph are $(0,2)$ and $(3,0)$, the vertical asymptote is defined by $x=1$ and the graph has no horizontal asymptote and no oblique asymptote. The first derivative is positive if $x>0$ and $x\neq 1,3$, and is negative if $x<0$, so f is increasing on the intervals $[0,1)$ and $(1,\infty)$ and f is decreasing on the interval $(-\infty,0]$, with a local minimum at the intercept $(0,2)$ and no local maximum. The second derivative is positive if $x<1$ or $x>3$ and is negative if $0<x<1$, so the graph is concave up on the intervals $(-\infty,1)$ and $[3,\infty)$ and is concave down on the interval $(1,3]$, with a point of inflection at the intercept $(3,0)$. Below is a sketch of the graph of f, with the vertical asymptote drawn as a dashed line, and the points of interest emphasized.



Solution to Question 10. — If x denotes the width and y denotes the height (each measured in metres) of the region for the climate slogan, then $xy = 24$ and it is requred to minimize the quantity $z = (x+1)(y+\frac{2}{3}) = 24+\frac{2}{3}x+24/x+\frac{2}{3}$, in which x is positive. Now

$$\frac{dz}{dx} = \frac{2}{3} - 24x^{-2} = \frac{2}{3}x^{-2}(x^2-36),$$

which is negative if $0 < x < 6$ and positive if $x > 6$, so the minimum value of z occurs where $x = 6$ and $y = 4$. Thus, the region for the climate slogan should be six metres wide and four metres tall to minimize the amount of cardboard used.

Solution to Question 11. — If $f(x) = (x-32)\sqrt[3]{x} = x^{4/3} - 32x^{1/3}$ then

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{32}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-8),$$

so the only critical number of f in the interval $(-8,1)$ is 0. Since

$$f(-8) = 80, \qquad f(0) = 0 \qquad \text{and} \qquad f(1) = -31,$$

the largest and smallest values of f on the given interval are, respectively, 80 and -31.

Solution to Question 12. — The velocity of the particle is $v = e^t - 4\cos(t) - 2\sin(t) + 5$, and the position of the particle is $s = e^t - 4\sin(t) + 2\cos(t) + 5t - 3$.

Solution to Question 13. — a. Expanding and integrating gives gives

$$\int \frac{(\sqrt{x}-3)(2x-1)}{\sqrt{x^3}}\,dx = \int \Big(2 - x^{-1} - 6x^{-1/2} + 3x^{-3/2}\Big)\,dx$$
$$= 2x - \log(x) - 12\sqrt{x} - 6\sqrt{x}^{-1}.$$

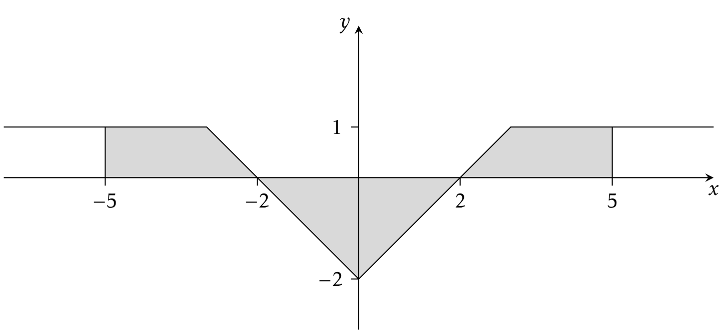
b. Integrating by inspection gives

$$\int_2^4 \Big\{\Big(\frac{2}{y}\Big)^2 + e^2\Big\}\,dy = \&Bigl\{-\frac{4}{y} + e^2y\Big\}\Big|_2^4$$
$$= -4\Big(\frac{1}{4} - \frac{1}{2}\Big) + e^2(4-2) = 1 + 2e^2.$$

c. Dividing and integrating by inspection gives

$$\int \frac{5\sin(x) - e^x\cos^2(x)}{\cos^2(x)}\,dx = \int \Big\{5\frac{\sin(x)}{\cos^2(x)} - e^x\Big\}\,dx = \frac{5}{\cos(x)} - e^x.$$

Solution to Question 14. — Below is a sketch of the graph of f, on the interval $[-7,7]$, with the region between the graph of f an the x axis shaded, for $-5 \leqslant x \leqslant 5$.



Since f is an even function, $\displaystyle\int_{-5}^{5} f = 2\int_0^5 f$, and (by considerations of oriented area)

$$\int_0^5 f = \int_0^3 f + \int_3^5 f = \tfrac{1}{2}(-2+1)(3-0) + 1\cdot(5-3) = \tfrac{1}{2},$$

it follows that

$$\int_{-5}^{5} f = 1.$$

Solution to Question 15. — If the interval $[0,5]$ is divided into k subintervals of equal length, then the length of each subinterval is $\frac{5}{k}$, the endpoints of the subintervals are $\frac{5j}{k}$, and the values of the integrand at these endpoints are $4 - \frac{25j^2}{k^2}$, for $j = 0,1,2,\ldots,k$. The corresponding Riemann sum obtained by evaluating the integrand at the right endpoint of each subinterval is

$$\mathscr{R}_k = \frac{5}{k}\sum_{j=1}^{k}\Big(4 - \frac{25}{k^2}j^2\Big) = 20 - \frac{125}{k^3}\sum_{j=1}^{k} j^2 = 20 - \frac{125}{k^3}\cdot\frac{1}{6}k(k+1)(2k+1)$$
$$= 20 - \frac{125}{3}\Big(1+\frac{1}{k}\Big)\Big(1+\frac{1}{2k}\Big).$$

Therefore,

$$\int_0^5 (4-x^2)\,dx = \lim_{k\to\infty}\mathscr{R}_k = 20 - \frac{125}{3} = -\frac{65}{3}.$$

Solution to Question 16. — After interchanging the limits of integration, Barrow's formula and the chain rule give

$$F'(x) = -\frac{d}{dx}\int_{\pi}^{e^{2x}} y\tan(y)\,dy = -e^{2x}\tan(e^{2x})\cdot 2e^{2x} = -2e^{4x}\tan(e^{2x}).$$

Solution to Question 17. — The polynomial defined by $p(x) = x^4 - x^3 - x - 4$ is continuous on $\mathbb{R}$, so the intermediate value theorm applies to p on any closed interval of positive length. Now $p(-2)p(-1) = 22\cdot(-1) < 0$ and $p(1)p(2) = (-5)\cdot 2 < 0$, so the intermediate value theorem implies that p has at least one zero in each of the intervals $(-2,-1)$ and $(1,2)$. Thus, the equation in question has at least one positive root, and at least one negative root.