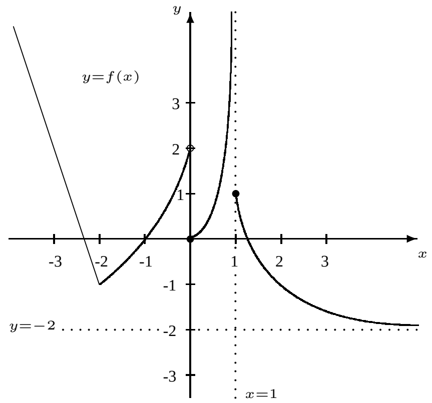
CALCULUS I (SCIENCE)

(MATHEMATICS 201–NYA/∞)

1. Evaluate the limit: $\lim\limits_{x\to 0} \dfrac{1-e^{2x^2}}{x^2}$.

   (Hint: After a change of variable the limit can be recognized as the value of a derivative. Any suggestion that the evaluation of this limit *requires* material from Calculus II reflects little more than an unfortunate incomprehension of the limit in question.)

2. Refer to the sketch below to evaluate the following limits. If a value does not exist, state in which way (∞, $-\infty$, or "does not exist").



   (a) $\lim\limits_{x\to -\infty} f(x)$
   (b) $\lim\limits_{x\to -2} f(x)$
   (c) $\lim\limits_{x\to 0} f(x)$
   (d) $\lim\limits_{x\to 1^-} f(x)$
   (e) $\lim\limits_{x\to 1^+} f(x)$
   (f) $\lim\limits_{x\to \infty} f(x)$

3. Calculate the following limits (if they exist). Make your answer as informative as possible: if a limit does not exist, say so; if appropriate one-sided limits exist instead, state them explicitly; if any limits are infinite, state this explicitly as well.

   (a) $\lim\limits_{x\to\infty} \dfrac{2x+7}{5x-x^2}$ (b) $\lim\limits_{x\to -3^-} \dfrac{|x+3|}{x+3}$ (c) $\lim\limits_{x\to 0} \dfrac{\sqrt{2x+4}-2}{x}$

   (d) $\lim\limits_{x\to 2} \dfrac{2x^2-5x+2}{x^2-4}$ (e) $\lim\limits_{x\to 1^-} \dfrac{x^2+1}{1-x}$

4. Find all values of c, if any, for which $f(x) = \begin{cases} cx+1 & \text{if } x \leqslant 3 \\ cx^2-1 & \text{if } x > 3 \end{cases}$ is continuous at $x=3$.

5. Sketch, if possible, the graph of a function that is continuous but not differentiable at $x=2$. If this is not possible, explain why.

6. State a limit definition of the derivative. Use this definition to find the derivative of $f(x) = x^2 - 3x$.

7. For each of the following functions, calculate the derivative $\dfrac{dy}{dx}$. Do not simplify your answers.

   (a) $y = 7x^3\sqrt{x} + \dfrac{5}{x} + \sqrt[6]{x^5} - 4\pi$ (b) $y = \dfrac{e^{x^3-x}}{2+7x^2}$

   (c) $y = 2\cos x - \sqrt{9-\sin^2 x}$ (d) $y = e^{\sec x}\tan(2x)$

   (e) $y = \left(x + e^{2x}\right)^{5x}$

8. For the function $f(x) = x\sqrt{98-x^2}$:
   (a) find $f'(x)$ and simplify your answer;
   (b) find the values of x for which the tangent line is horizontal.

9. Find an equation for the line tangent to the graph of $y = \dfrac{8}{\sqrt{4+3x}}$ at the point $(4,2)$.

10. Given $xe^{x-y^2} = x^2 - y^2$, find the slope of the tangent line to the curve at $(2,\sqrt{2})$.

11. Given $y = \ln(1+x^2)$, find and simplify the second derivative y'' or $\dfrac{d^2y}{dx^2}$.

12. The position (in metres) of a particle at time t (seconds) is given by the equation
$$x = \frac{t^2}{200} + \ln(t+1).$$
Find the velocity when the acceleration is 0. (Assume $t \geqslant 0$.)

13. Evaluate the following integrals:

    (a) $\displaystyle\int \frac{x^5+2x-\sqrt{3x}}{x^4}\,dx$ (b) $\displaystyle\int \left(\frac{2}{t} + e^t - \cos t\right) dt$

    (c) $\displaystyle\int \left(2+x^2\right)^2 dx$ (d) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\varphi + \frac{2}{\sin^2\varphi}\right) d\varphi$

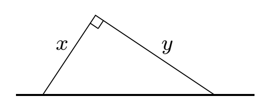
14. Given that $f'(x) = 2-4x$, and $f(1) = 5$, find $f(x)$.

15. Find the area of the region which lies between the curve $y = 16 - x^4$ and the x-axis.

16. An object is moving with an acceleration given by the equation $a(t) = 16 - t^2$. (t is time in seconds, $t \geqslant 0$, a is acceleration in $\mathrm{m/s^2}$.) At what time is the velocity of the object maximal?

17. Determine whether or not the function
$$S(x) = \frac{1}{x} + x^2$$
has a maximum value. If it does, what is the maximum value? If you think it does not have a maximum, justify your claim.

18. An enclosure in the form of a right-angled triangle is constructed using some fencing along two-sides, and the river bank along the hypotenuse, as shown in the diagram. If 400 m of fencing is available, what are the dimensions x and y that maximize the area of the enclosure?



19. Find the vertical and horizontal asymptotes to the graph of
$$f(x) = \left(\frac{2x-77}{4+300x}\right)^2.$$

20. For the function $f(x) = x\sqrt{1-x^2}$, the first and second derivatives are
$$f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}} \quad\text{and}\quad f''(x) = \frac{x(2x^2-3)}{(1-x^2)^{3/2}}.$$
    (a) Find the intervals where the function is increasing, and the intervals where it is decreasing.
    (b) Find the intervals where the function is concave up, and the intervals where it is concave down.
    (c) Find the coordinates of all relative (or local) extreme points.
    (d) Find the coordinates of all points of inflection.
    (e) Sketch the graph of f.

    Make sure that your graph clearly illustrates all these features.