CEGEP JOHN ABBOTT COLLEGE

# CALCULUS II

(MATHEMATICS 201–NYA-05)



1. Let $y = \arcsin(1/x) + \operatorname{arcsec} x$, for $x > 1$. Find $dy/dx$ and simplify your answer.

2. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{x+21}{(x+1)(x^2+9)}\,dx$ (b) $\displaystyle\int e^{2x}\cos 6x\,dx$

   (c) $\displaystyle\int \frac{dx}{(4x^2-9)^{3/2}}$ (d) $\displaystyle\int \ln(x^2+1)\,dx$

   (e) $\displaystyle\int \frac{\sin^2\sqrt{x}\,\cos^3\sqrt{x}}{\sqrt{x}}\,dx$ (f) $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{\sec^2 x\,dx}{\sqrt{4-\tan^2 x}}$

3. Evaulate each of the following limits.

   (a) $\displaystyle\lim_{x\to 0^+} \frac{\ln\sin x}{\ln\tan x}$ (b) $\displaystyle\lim_{x\to 0}(1-2x)^{2/x}$

   (c) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x^2} - \frac{1}{\sin x}\right)$

4. Evaluate each of the following improper integrals.

   (a) $\displaystyle\int_0^{\infty}(x-1)e^{-x}\,dx$ (b) $\displaystyle\int_3^5 \frac{2x-3}{x-3}\,dx$

5. (a) Sketch the region $\mathscr{R}_1$ bounded by the parabola $y = x^2+1$ and the line $y = x+3$. Find the area of $\mathscr{R}_1$.

   (b) Sketch the region $\mathscr{R}_2$ enclosed by $y = \sin x$ and the $x$-axis, from $x = 0$ to $x = \pi$. Set up an integral that represents the volume of the solid obtained by revolving $\mathscr{R}_2$ about

   (i) the $x$-axis,

   (ii) the $y$-axis.

   Evaluate *one* of these integrals.

6. Solve the initial-value problem:

$$\frac{dy}{dx} = \frac{y+1}{\cos x}, \quad y\left(\tfrac{1}{4}\pi\right) = \sqrt{2}.$$

7. For each of the following, state whether the sequence $\{a_n\}$ converges and, if so, to what value.

   (a) $a_n = \dfrac{(2n)!}{(2n+2)!}$ (b) $a_n = \dfrac{\sin n}{n}$

   (c) $a_n = \ln(2n-1) - \ln(n+4)$

8. (a) Does the series

$$\sum_{n=1}^{\infty}\frac{4^n}{n!}$$

   converge or diverge? Justify your answer.

   (b) Does the sequence

$$a_n = \frac{4^n}{n!}$$

   converge or diverge? Justify your answer.

9. Determine whether or not each of the following series converges or diverges. Justify all assertions.

   (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(n+1)^2}{2n+3n^2}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+1}{n^5+n^4}$ (d) $\displaystyle\sum_{n=1}^{\infty}\frac{5^{n+1}-3^{n-1}}{4^n}$

10. State, with justification, whether each of the following series converges absolutely, converges conditionally, or diverges.

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+1)^2}{3n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n\sqrt{n}}$

   (c) $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$ (d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-\sqrt{n}}}{\sqrt{n}}$

11. Find the Taylor series of $f(x) = \sin x$ centred at $\pi$. Express the series using sigma notation.

12. Find the sequence of partial sums of the series

$$\sum_{n=1}^{\infty}\left\{\ln(2n+3) - \ln(2n+1)\right\},$$

   and use it to determine whether the series converges.

13. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x+4)^n}{3^n\sqrt{n+1}}.$$

## ANSWERS

In questions 7–13, $a_n$ denotes the general term of the sequence or series in question. In question 12, $s_n$ denotes the general term of the sequence of partial sums of the series in question.

1. 0

2. (a) $\ln\dfrac{(x+1)^2}{x^2+9} + \arctan\left(\tfrac{1}{3}x\right) + C$, (b) $\tfrac{1}{20}e^{2x}(3\sin 6x + \cos 6x) + C$,

   (c) $\dfrac{-x}{9\sqrt{4x^2-9}} + C$, (d) $x\ln(x^2+1) - 2x + 2\arctan x + C$,

   (e) $\tfrac{2}{3}\sin^3\sqrt{x} - \tfrac{2}{5}\sin^5\sqrt{x} + C$, (f) $\tfrac{1}{6}\pi$.

3. (a) 1, (b) $e^{-4}$, (c) $\infty$ (so the limit does not exist).

4. (a) 0, (b) $\infty$ (so the integral diverges).

5. (a) $\tfrac{9}{2}$,

   (b) (i) $\displaystyle\pi\int_0^{\pi}\sin^2 x\,dx = \tfrac{1}{2}\pi^2$, (ii) $\displaystyle 2\pi\int_0^{\pi} x\sin x\,dx = 2\pi^2$.

6. $y = \sec x + \tan x - 1$.

7. (a) $a_n \to 0$, (b) $a_n \to 0$, and (c) $a_n \to \ln 2$, each as $n\to\infty$.

8. (a) $\sum a_n$ converges by the ratio test. (b) $a_n \to 0$, by (a) and the vanishing criterion.

9. (a) $\sum a_n$ converges by the root test ($\sqrt[n]{a} \to \tfrac{1}{3}$ as $n\to\infty$).
   (b) $\sum a_n$ diverges by the ratio test ($a_{n+1}/a_n \to e$ as $n\to\infty$).
   (c) $\sum a_n$ converges by the comparison test ( $|a_n| < 2n^{-2}$).
   (d) $\sum a_n$ diverges by the vanishing criterion ($a_n \to \infty$ as $n\to\infty$).

10. (a) $\sum a_n$ diverges by the vanishing criterion ($|a_n| \to \tfrac{1}{3}$ as $n\to\infty$).
   (b) $\sum a_n$ is absolutely convergent by the comparison test ($|a_n| < n^{-3/2}$).
   (c) $\sum a_n$ is conditionally convergent by the alternating series test ($|a_n| \downarrow 0$ as $n\to\infty$) and the comparison test ($|a_n| > n^{-1}$ if $n \geqslant 3$).
   (d) $\sum a_n$ is absolutely convergent by the comparison test ($|a_n| < n^{-3/2}$).

11. $\displaystyle\sin x = -\sin(x-\pi) = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}(x-\pi)^{2n+1}}{(2n+1)!}$

12. $\sum a_n$ diverges since $s_n = \ln\left\{\tfrac{1}{3}(2n+3)\right\} \to \infty$ as $n\to\infty$.

13. The interval of convergence of the power series is $(\,-7,-1\,]$.

Hereafter, according to the context, $\mathscr{I}$ denotes the integral in question, $\ell$ denotes the limit in question, $A$ denotes the area in question, $V$ denotes the volume in question, $a_n$ denotes the general term of the sequence or series in question and $s_n$ denotes the general term of the sequence of partial sums of the series in question. The symbol $\overset{\ell'HR}{=}$ indicates an application of l'Hôpital's rule.

1. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}\dfrac{1}{\sqrt{1-1/x^2}} + \dfrac{1}{x\sqrt{x^2-1}} = 0.$

2. (a) $\mathscr{I} = \displaystyle\int \frac{2\,dx}{x+1} - \int \frac{2x-3}{x^2+9}\,dx = \ln\frac{(x+1)^2}{x^2+9} + \arctan\big(\tfrac13 x\big) + C$

(b) $\mathscr{I} = \frac{1}{18}e^{2x}(3\sin 6x + \cos 6x) - \frac19\mathscr{I}$

$\therefore\ \mathscr{I} = \frac{1}{20}e^{2x}(3\sin 6x + \cos 6x) + C$

| | | |
|---|---|---|
| $+$ | $e^{2x}$ | $\cos 6x$ |
| $-$ | $2e^{2x}$ | $\frac16 \sin 6x$ |
| $+$ | $4e^{2x}$ | $-\frac{1}{36}\cos 6x$ |

(c) $\mathscr{I} = \displaystyle\int \frac{dt}{9t^2} = -\frac{1}{9t} + C = \frac{-x}{9\sqrt{4x^2-9}} + C$, where $t^2 = 4 - 9x^{-2}$.

(d) $\mathscr{I} = x\ln(x^2+1) - \displaystyle\int \frac{2x^2}{x^2+1}\,dx = x\ln(x^2+1) - 2x + 2\arctan x + C$

(e) $\mathscr{I} = \displaystyle\int 2t^2(1-t^2)\,dt = \tfrac23\sin^3\sqrt{x} - \tfrac25\sin^5\sqrt{x} + C$, where $t = \sin\sqrt{x}$.

(f) $\mathscr{I} = \displaystyle\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \arcsin\big(\tfrac12 t\big)\Big|_0^1 = \tfrac16\pi$, where $t = \tan x$.

3. (a) $\ell \overset{\ell'HR}{=} \displaystyle\lim_{x\to0^+} \frac{\cot x \tan x}{\sec^2 x} = 1$

(b) $\ell = \displaystyle\lim_{x\to0} e^{2\ln(1-2x)/x} \overset{\ell'HR}{=} \lim_{x\to0} e^{-4/(1-2x)} = e^{-4}$

(c) $\ell = \displaystyle\lim_{x\to0^+} \frac1x\left(\frac1x - \frac{x}{\sin x}\right) = \infty$

4. (a) $\mathscr{I} = \displaystyle\lim_{t\to\infty}\big(-xe^{-x}\big)\Big|_0^t \overset{\ell'HR}{=} 0$ (b) $\mathscr{I} = \displaystyle\lim_{t\to3^+}\big(2x + 3\ln(x-3)\big)\Big|_t^5 = \infty$

5. (a) $A = \displaystyle\int_{-1}^2 \big\{(x+3)^2 - (x^2+1)^2\big\}\,dx = \tfrac92$

(b) (i) $V = \pi\displaystyle\int_0^\pi \sin^2 x\,dx = \tfrac12\pi(x - \sin x\cos x)\Big|_0^\pi = \tfrac12\pi^2$

(ii) $V = 2\pi\displaystyle\int_0^\pi x\sin x\,dx = 2\pi(\sin x - x\cos x)\Big|_0^\pi = 2\pi^2$

6. One has $\displaystyle\int\frac{dy}{y+1} = \int\frac{dx}{\cos x}$, so $y = A(\sec x + \tan x) - 1$. Using $y\big(\tfrac14\pi\big) = \sqrt2$, gives $y = \sec x + \tan x - 1$.

7. (a) $a_n = \dfrac{1}{2(n+1)(2n+1)} \to 0$ as $n\to\infty$.

(b) $|a_n| < 1/n \to 0$, and therefore $a_n \to 0$ as $n\to\infty$ by the squeeze theorem.

(c) $a_n = \ln\dfrac{2-1/n}{1+4/n} \to \ln 2$ as $n\to\infty$.

8. (a) $a_n > 0$ and $a_{n+1}/a_n = 4/(n+1) \to 0$ as $n\to\infty$, so $\sum a_n$ converges by the ratio test.

(b) $a_n \to 0$, *e.g.*, by the vanishing criterion.

9. (a) $\sqrt[n]{a_n} = \dfrac{(1+1/n)^2}{2/n+3} \to \frac13$ as $n\to\infty$, so $\sum a_n$ converges by the root test.

(b) $a_{n+1}/a_n = \big(1+\frac1n\big)^n \to e$ as $n\to\infty$, so $\sum a_n$ diverges by the ratio test.

(c) $0 < a_n \leqslant 2n^{-2}$, so $\sum a_n$ converges with $\sum n^{-2}$ by the comparison test.

(d) $a_n\to\infty$ as $n\to\infty$ so $\sum a_n$ diverges by the vanishing criterion.

10. (a) $|a_n| \to \frac13$ as $n\to\infty$ so $\sum a_n$ diverges by the vanishing criterion.

(b) $|a_n| < n^{-3/2}$ so $\sum a_n$ is absolutely convergent by the comparison test.

(c) $|a_n|$ is decreasing if $n \geqslant 3$ and $|a_n| \to 0$ (*e.g.*, by l'Hôpital's rule) so $\sum a_n$ is convergent; but $|a_n| > n^{-1}$ if $n\geqslant 3$ so $\sum|a_n|$ diverges by the comparison test (with the harmonic series). Therefore, $\sum a_n$ is conditionally convergent.

(d) The maximum value of the function $f(x) = \ln x/\sqrt{x}$ is $2/e$ (at $x = e^2$), so $e^{-\sqrt{n}} < n^{-1}$. Therefore $|a_n| < n^{-3/2}$, so $\sum a_n$ is absolutely convergent by the comparison test.

11. $\sin x = -\sin(x-\pi) = \displaystyle\sum_{n=0}^\infty \frac{(-1)^{n-1}(x-\pi)^{2n+1}}{(2n+1)!}$

12. $s_n = \ln\big\{\frac13\ln(2n+3)\big\} \to \infty$ as $n\to\infty$, so $\sum a_n$ diverges.

13. $|a_{n+1}/a_n| \to \frac13|x+4|$ as $n\to\infty$, so the radius of convergence of $\sum a_n$ is 3. When $x = -7$ the series diverges because it is (a tail of) the $p$-series with $p = \frac12$, and when $x = -1$ the series converges by the alternating series test. Therefore, the interval of convergence of the power series is $(-7,-1]$.