CALCULUS II (SCIENCE)

(MATHEMATICS 201-NYB-05)

1. Find the derivative with respect to x of
$$y = (\arctan x)^2 + x\arctan x - \operatorname{arcsec}\sqrt{x}.$$

2. Calculate the following limits.
   a. $\displaystyle\lim_{x\to\infty}\left(xe^{1/x} - x\right)$ b. $\displaystyle\lim_{x\to 0^+}(1+2x)^{1/(3x)}$

3. Evaluate the following integrals.
   a. $\displaystyle\int \frac{(4+\sqrt{x})3^{\sqrt{x}+2\ln x}}{2x}\,dx$ b. $\displaystyle\int \frac{\sec^3(\ln x)\tan^3(\ln x)}{x}\,dx$
   c. $\displaystyle\int \frac{x+2}{\sqrt{2x-1}}\,dx$ d. $\displaystyle\int \frac{x^2+3}{x^3+2x}\,dx$
   e. $\displaystyle\int e^{2x}\sin 4x\,dx$ f. $\displaystyle\int_3^{3\sqrt{2}} \frac{dx}{x^3\sqrt{x^2-9}}$

4. Determine whether the improper integrals converge or diverge; if an integral converges, give its exact value.
   a. $\displaystyle\int_1^2 \frac{3\,dx}{\sqrt{4-x^2}}$ b. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

5. Compute the exact area of the region bounded by the curves $y = 4 - x^2$ and $y = x^2 + 2$ between $x = 0$ and $x = 2$. Sketch the region.

6. a. Sketch the region $\mathscr{S}$ bounded by $y = 4 - x^2$ and $y = x^2 + 2$ on the interval $[1, 2]$.
   b. Find the volume of the solid obtained when $\mathscr{S}$ is revolved about i. the line $x = 1$, ii. the x-axis.

7. Solve the differential equation
$$\frac{\sqrt{1-x^2}}{y}\frac{dy}{dx} + x = 0; \quad y(0) = 1.$$
Express your final answer as an explicit function of x.

8. Determine, with justification, whether the sequence converges or diverges. If a sequence converges, give the value to which it converges.
   a. $\left\{\dfrac{(\ln n)^3}{n}\right\}$ b. $\left\{\dfrac{e^{3/n}}{3n}\right\}$ c. $\left\{(-1)^n\cos^2\pi n\right\}$

9. Determine whether the series converges or diverges. If a series converges, give its sum.
   a. $\displaystyle\sum_{n=2}^{\infty}\left\{\operatorname{arcsec} n - \operatorname{arcsec}(n+1)\right\}$ b. $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{e^n} + \frac{2}{3^n}\right\}$

10. Classify each series as convergent or divergent. State the test you use, and verify the conditions for using the test are satisfied.
   a. $\displaystyle\sum_{n=1}^{\infty}\left\{4+\frac{2}{n}\right\}$ b. $\displaystyle\sum_{n=1}^{\infty}\left\{\sqrt[n]{3}-1\right\}^n$
   c. $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{1}{n}-\frac{1}{n^2}\right\}$ d. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 2^n}{(2n)!}$

11. Classify each series as absolutely convergent, conditionally convergent, or divergent. Justify your conclusions.
   a. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sin^2\frac{1}{2}\pi n}{n^3}$ b. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$

12. Determine the interval of convergence of the series
$$\sum_{n=1}^{\infty}\frac{2^{n-1}(x-1)^n}{(n+1)!}.$$

13. a. Use a known power series to determine the sum of the following series.
   i. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ ii. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}$
   b. Use a known power series to express $f(x) = xe^{-2x}$ as a power series centred at 0. What is the interval of convergence of the resulting series.

14. For the function $f(x) = \ln(x+1)$:
   a. find the first five non-zero terms of the Maclaurin series of $f(x)$;
   b. find the n^{th} term of the series, and write the series using sigma notation;
   c. determine the radius of convergence of its Maclaurin series.

ANSWERS

In Question 8, a_n denotes the general term of the sequence in question. In Questions 10 and 11, a_n denotes the general term of the series in question.

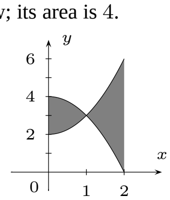
1. $\dfrac{dy}{dx} = \dfrac{2\arctan x}{1+x^2} + \dfrac{x}{x^2+1} + \arctan x - \dfrac{1}{2x\sqrt{x-1}}$.

2. a. 1, b. $e^{2/3}$.

3. a. $(3^{\sqrt{x}+2\ln x})/\ln 3 + C$, b. $\frac{1}{5}\sec^5(\ln x) - \frac{1}{3}\sec^3(\ln x) + C$, c. $\frac{1}{3}(x+7)\sqrt{2x-1} + C$, d. $\frac{1}{6}\ln\dfrac{x^6}{x^2+2} + C$,
   e. $\frac{1}{10}e^{2x}(\sin 4x - 2\cos 4x) + C$, f. $(\pi+2)/216$.

4. Each of the integrals converges to π.

5. $\mathscr{R}$ is the shaded region below; its area is 4.



6. a. $\mathscr{S}$ is the portion of $\mathscr{R}$ (see the answer to the previous problem) to the right of the line $x = 1$ (where the curves meet).
   b. i. The volume is given by $2\pi\displaystyle\int_1^2 (x-1)(2x^2-2)\,dx = \frac{11}{3}\pi$.
   ii. The volume is given by $\pi\displaystyle\int_1^2 (12x^2-12)\,dx = 16\pi$.

7. $y = e^{\sqrt{1-x^2}-1}$.

8. a. $\lim a_n = 0$. b. $\lim a_n = 0$. c. $\{a_n\}$ diverges.

9. a. $-\frac{1}{6}\pi$, b. $e/(e+1)$.

10. a. $\sum a_n$ diverges by the Vanishing Criterion ($\lim a_n = 4$).
   b. $\sum a_n$ converges by the Root Test ($\lim\sqrt[n]{|a_n|} = 0$).
   c. $\sum a_n$ diverges by the Comparison Test, because $a_n \geqslant \frac{1}{2}n^{-1}$ if $n \geqslant 2$, and the harmonic series diverges.
   d. $\sum a_n$ converges by the Ratio Test ($\lim|a_{n+1}/a_n| = 0$).

11. a. $|a_n| \leqslant n^{-3}$, and $\sum n^{-3}$ is a convergent p-series, so $\sum a_n$ is absolutely convergent by the Comparison Test.
   b. $\sum a_n$ is conditionally convergent (convergent by the Alternating Series Test, and not absolutely convergent by the Condensation Test).

12. The interval of convergence of the given series is $\mathbb{R}$.

13. a. The sum of the series is equal to $\arctan 1 = \frac{1}{4}\pi$.
   b. The sum of the series is equal to $\cos\frac{1}{6}\pi = \frac{1}{2}\sqrt{3}$.

14. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$.
   The radius of convergence of the Maclaurin series is 1.