CALCULUS II (SCIENCE)

(MATHEMATICS 201-NYB-05)

1. Differentiate
$$y = \tfrac{1}{2}x\sqrt{9-4x^2} + \tfrac{9}{4}\arcsin\big(\tfrac{2}{3}x\big)$$
with respect to x, and simplify your answer as much as possible.

2. Evaluate each of the following integrals. Give all answers in simplified exact form, without using decimals.

   a. $\displaystyle\int (2t+1)\sqrt{t+1}\,dt$ b. $\displaystyle\int e^{3x}\sin 4x\,dx$

   c. $\displaystyle\int \frac{\sin^3(1/x)\cos^2(1/x)}{x^2}\,dx$ d. $\displaystyle\int \frac{11x^2-14x+8}{(x^2+1)(2x-1)}\,dx$

   e. $\displaystyle\int \frac{dx}{x^3\sqrt{4x^2-9}}$ f. $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{\sec x\tan x}{\sqrt{4-\sec^2 x}}\,dx$

3. Determine the value of each limit, and justify your answers by showing all relevant steps. Use proper mathematical notation throughout.

   a. $\displaystyle\lim_{x\to 0}\frac{\sin^2 2x}{\cos^3 3x - 1}$ b. $\displaystyle\lim_{x\to 1^+} x^{2/(x-1)}$

4. Determine whether the improper integrals converge or diverge; if an integral converges, give its exact value. Use correct mathematical notation throughout.

   a. $\displaystyle\int_{-\infty}^{2}\frac{3\,dx}{x^2+4}$ b. $\displaystyle\int_1^3 \frac{3\,dx}{x^2-x-2}$

5. Find the area of the region bounded by the graphs of $y = x^2$ and $y = \sqrt{8x}$. Give your answer in simplified exact form, without using decimal fractions.

6. Let $\mathscr{R}$ be the region between the graphs of
$$y = \frac{3}{x} \qquad \text{and} \qquad y = \frac{3x}{x^2+1},$$
from $x = 1$ to $x = \sqrt{3}$.

   a. Set up, but do not evaluate, an integral that represents the volume of the solid generated by revolving $\mathscr{R}$ about the x-axis.

   b. Find the volume of the solid generated by revolving $\mathscr{R}$ about the y-axis. Give your answer in simplified exact form, without using decimals.

7. Solve the differential equation $dy/dx = y\sin x$, subject to $y\big(\tfrac{1}{2}\pi\big) = 2$. In your final answer, express y as an explicit function of x.

8. Determine whether the sequence converges or diverges. Justify your answer. If a sequence converges, give its limit. If a sequence diverges, explain why it diverges. Give all answers in simplified exact form, without using decimals.

   a. $a_n = \big(1 - \frac{1}{3n}\big)^{5n}$ b. $b_n = \big(\frac{1}{3n} - 1\big)^{5n}$

9. Find the sum of the series. Give all answers in simplified exact form, without using decimals.

   a. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+2n}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+4^{n+1}}{5^n}$ c. $\displaystyle\sum_{n=0}^{\infty}\frac{\sin^n x}{3^n}$

10. Determine whether the series converges or diverges. Justify your answers completely. (State, and verify explicitly the hypotheses, of each test you use.)

    a. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{3^n n!}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n+\cos^2 n}$

    c. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3k^3+1}{2k^3+1}\right)^k$ d. $\displaystyle\sum_{n=2}^{\infty}\sin\big(1/n^2\big)$

11. Determine whether each series is absolutely or conditionally convergent, or divergent.

    a. $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$ b. $\displaystyle\sum_{k=0}^{\infty}(-2)^k\frac{k+1}{5^k}$

12. Determine the interval of convergence of the power series
$$\sum_{n=1}^{\infty}\frac{2n-1}{2^n(n+1)}(x-2)^n.$$

13. Find the the Taylor series of $\sqrt{x+1}$ centred at 3, and write out the first four terms of this series explicitly.

ANSWERS

1. $dy/dx = \sqrt{9-4x^2}$.

2. a. $\frac{2}{15}(6t+1)(t+1)^{3/2} + C$, b. $\frac{1}{25}e^{3x}(3\sin 4x - 4\cos 4x) + C$

   c. $\frac{1}{3}\cos^3(1/x) - \frac{1}{5}\cos^5(1/x) + C$,

   d. $\frac{1}{2}\ln|(2x-1)^3(x^2+1)^4| - 5\arctan x + C$,

   e. $\sqrt{4x^2-9}/(18x^2) + \frac{2}{27}\arctan\frac{1}{3}\sqrt{4x^2-9} + C$, f. $\frac{1}{12}\pi$.

3. a. $-\frac{8}{27}$ (this is a Cal I limit), b. e^2 (by the definition of the derivative).

4. a. $\frac{9}{8}\pi$, b. the integral is divergent.

5. The area of the region is $\displaystyle\int_0^2(\sqrt{8x}-x^2)\,dx = \tfrac{8}{3}$.

6. a. The volume of the solid is
$$9\pi\int_1^{\sqrt{3}}\left\{\frac{1}{x^2} - \frac{x^2}{(x^2+1)^2}\right\}dx \quad \big(= \tfrac{27}{4}\pi - \tfrac{15}{8}\sqrt{3}\pi - \tfrac{3}{8}\pi^2\big).$$
   b. The volume of solid is
$$6\pi\int_1^{\sqrt{3}}\left\{1 - \frac{x^2}{x^2+1}\right\}dx = 6\pi\arctan x\,\Big|_1^{\sqrt{3}} = \tfrac{1}{2}\pi^2.$$

7. $y = 2e^{-\cos x}$.

8. a. $\lim a_n = e^{-5/3}$,

   b. $\lim b_{2n+1} = -e^{-5/3}$ and $\lim b_{2n} = e^{-5/3}$, so $\{b_n\}$ diverges.

9. a. $\frac{3}{4}$, b. $\frac{35}{2}$, c. $3/(3-\sin x)$.

10. In this question, let a_n (or a_k as appropriate) denote the general term of the series in question.

    a. $\sum a_n$ converges by the Ratio Test ($\lim a_{n+1}/a_n = \frac{1}{3}e$).

    b. $0 < a_n < 3^{-n}$, and $\sum 3^{-n}$ is a convergent geometric series, so $\sum a_n$ converges by the Comparison Test.

    c. $\sum a_k$ diverges by the Root Test ($\lim \sqrt[k]{a_k} = \frac{3}{2}$).

    d. $0 < a_n < n^{-2}$ for $n \geqslant 1$, and $\sum n^{-2}$ is a convergent p-series, so $\sum a_n$ converges by the Comparison Test.

11. In this question, let $(-1)^k a_k$ denote the general term of the alternating series in question.

    a. $\{a_k\}$ is positive, decreasing, and converges to 0, so $\sum(-1)^k a_k$ converges by the Alternating Series Test. On the other hand, $\sum a_k$ is a divergent logarithmic p-series (by the Condensation Test, or the Integral Test), so $\sum(-1)^k(k\ln k)^{-1}$ is conditionally convergent.

    b. Since $\lim a_{k+1}/a_k = \frac{2}{5}$, $\sum(-1)^k a_k$ is absolutely convergent by the Ratio Test.

12. The interval of convergence of the given series is $(\,0, 4\,)$.

13.
$$\begin{aligned}\sqrt{1+x} &= \sqrt{4+(x-3)} = 2\sqrt{1+(x-3)/4}\\ &= 2 + 2\sum_{n=1}^{\infty}\frac{\frac{1}{2}\big(-\frac{1}{2}\big)\cdots\big(\frac{1}{2}-n+1\big)}{2^{2n}n!}(x-3)^n\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n)!}{2^{4n-1}(n!)^2(2n-1)}(x-3)^n\\ &= 2 + \tfrac{1}{4}(x-3) - \tfrac{1}{64}(x-3)^2 + \tfrac{1}{512}(x-3)^3 - \cdots.\end{aligned}$$