CALCULUS II (SCIENCE)

(MATHEMATICS 201-NYB-05)

1. Differentiate $y = 2x \operatorname{arcsec} x - \sqrt{x^2-1}$ with respect to x, and simplify your answer.

2. Evaluate the integrals.

   a. $\displaystyle\int_0^{1/4} \frac{\arccos 2x}{\sqrt{1-4x^2}}\,dx$ b. $\displaystyle\int \frac{(x+2)}{\sqrt{2x+1}}\,dx$ c. $\displaystyle\int x\arctan x\,dx$

   d. $\displaystyle\int \tan^3 \tfrac{1}{2}x\,dx$ e. $\displaystyle\int_0^{\pi/4} \sin^3 2x\cos^2 2x\,dx$

   f. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$ g. $\displaystyle\int \frac{3x^2-2x+9}{(x-1)(x^2+4)}\,dx$

3. Evaluate the improper integrals.

   a. $\displaystyle\int_1^{\infty} \frac{1}{x^2+2x}\,dx$ b. $\displaystyle\int_{-1}^{2} \frac{(x+1)}{[x(x+2)]^{4/3}}\,dx$

4. Evaluate the limits.

   a. $\displaystyle\lim_{x\to+\infty}\left(\frac{x+2}{x+3}\right)^x$ b. $\displaystyle\lim_{x\to 1}\frac{x-e^{x-1}}{(x-1)^2}$ c. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x\cos x}\right)$

5. Let $\mathscr{S}$ be the region bounded by the graphs of $x = y - y^2$ and $x = 0$.

   a. Compute the area of the region $\mathscr{S}$.
   b. Find the volume of the solid obtained when this region is rotated about the x-axis.
   c. Find the volume of the solid obtained when this region is rotated about the y-axis.

6. Let $\mathscr{R}$ be the region bounded by the graphs of

   $$y = x\sin x, \quad y=0, \quad x=0, \quad \text{and} \quad x=2.$$

   a. Set up the integrals required to compute the volume of the solid obtained by rotating $\mathscr{R}$ about i. the x-axis, and ii. the y-axis.
   b. Evaluate **one** of the integrals from part a.

7. Solve the differential equation:

   $$(x^2+1)\frac{dy}{dx} = y; \quad y(1) = 2.$$

8. Does the sequence

   $$\left\{\frac{3(n-1)!}{5(n+1)!}\right\}$$

   converge? If so, find its limit as $n\to\infty$. Justify your answer.

9. Determine whether the series

   $$\sum_{n=1}^{\infty}\left(\arccos\left(\tfrac{1}{n+1}\right) - \arccos\left(\tfrac{1}{n}\right)\right)$$

   converges or diverges; if it converges, find the sum. Justify your answer.

10. Determine whether each of the following series converges or diverges. State the tests you use, and verify that the conditions for using them are satisfied.

    a. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{n+2}{2n+1}\right)^n$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$

11. Label each series as absolutely convergent, conditionally convergent, or divergent. Justify your answers.

    a. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n(n+1)}{(n+2)(n+3)}$ b. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(n!)^2}{(2n)!}$

12. Find the radius and interval of convergence of the power series

    $$\sum_{n=0}^{\infty}\frac{3^n\,(x-2)^n}{n^3+1}.$$

13. Find the Taylor series of $f(x) = 1/x$ centered at 1.

ANSWERS

1. $\displaystyle 2\operatorname{arcsec} x - \frac{x-2}{\sqrt{x^2-1}}$, or $\displaystyle\frac{2\operatorname{arcsec} x\sqrt{x^2-1} - x + 2}{\sqrt{x^2-1}}$.

2. a. $\frac{5}{144}\pi^2$; b. $\frac{1}{3}(x+5)\sqrt{2x+1}+C$;
c. $\frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C$; d. $\tan^2\frac{1}{2}x + 2\ln|\cos\frac{1}{2}x| + C$;
e. $\frac{1}{15}$; f. $-\frac{1}{2}x^{-2}\sqrt{x^2-9} + \frac{1}{6}\arctan\sqrt{x^2-9} + C$;
g. $\frac{1}{2}\ln|(x-1)^4(x^2+4)| - \frac{1}{2}\arctan\frac{1}{2}x + C$.

3. a. The integral converges to $\frac{1}{2}\ln 3$. b. The integral diverges (to ∞).

4. a. $1/e$; b. $-\frac{1}{2}$; c. 0.

5. a. The area of $\mathscr{S}$ is $\frac{1}{6}$. b. The volume of the solid obtained when $\mathscr{S}$ is revolved about the x-axis is

$$2\pi\int_0^1 y(y-y^2)\,dy = \tfrac{1}{6}\pi.$$

c. The volume of the solid obtained when $\mathscr{S}$ is revolved about the y-axis is

$$\pi\int_0^1 (y-y^2)^2\,dy = \tfrac{1}{30}\pi.$$

6. i. The volume of the solid obtained by revolving $\mathscr{R}$ about the x-axis is

$$\pi\int_0^2 x^2\sin^2 x\,dx = \tfrac{1}{24}(32 - 21\sin 4 - 12\cos 4).$$

ii. The volume of the solid obtained by revolving $\mathscr{R}$ about the y-axis is

$$2\pi\int_0^2 x^2\sin x\,dx = 4\pi(2\sin 2 - \cos 2 - 1).$$

7. $y = e^{\arctan x - \pi/4}$.

8. $\displaystyle\frac{3(n-1)!}{5(n+1)!} = \frac{3}{5n(n+1)} \to 0$, as $n\to\infty$, so the given sequence converges to zero.

9. Let s_n denote the sum of the first n terms of the series in question. Then $s_n = \arccos\left(\frac{1}{n+1}\right) - \arccos 1 = \arccos\left(\frac{1}{n+1}\right) \to \frac{1}{2}\pi$, as $n\to\infty$. Therefore, the given series converges to $\frac{1}{2}\pi$.

10. In each part of this problem, a_n will denote the general term of the series in question.
a. $\sqrt[n]{a_n} = (n+2)/(2n+1) \to \frac{1}{2}$ (< 1) as $n\to\infty$, so $\sum a_n$ converges by the root test.
b. If $n \geqslant 3$ then $\ln n > 1$ and so $a_n > 1/n$. Therefore, $\sum a_n$ diverges with the harmonic series by the comparison test.
c. If $n > 1$ then $1/n < 1$, so $e^{1/n} < e$, and hence $0 < a_n < e/n^2$. Therefore, $\sum a_n$ converges with $\sum n^{-2}$ (which is a convergent p-series: $p = 2 > 1$) by the comparison test.

11. In each part of this problem, the general term of the series in question will be written as $(-1)^n a_n$, as given. Notice that in each case, $a_n \geqslant 0$ for all $n \geqslant 0$.
a. $\lim a_n = 1$ $(\neq 0)$, so $\sum(-1)^n a_n$ diverges by the vanishing criterion.
b. $|a_{n+1}/a_n| \to \frac{1}{4}$ (< 1) as $n\to\infty$, so $\sum(-1)^n a_n$ is absolutely convergent by the ratio test.

12. Let u_n denote the general term of the power series in question. Then $|u_{n+1}/u_n| \to 3|x-2|$, which is less that 1 if, and only if x belongs to $\left(\frac{5}{3}, \frac{7}{3}\right)$, so the series converges absolutely in this interval and its radius of convergence is $\frac{1}{3}$. At the endpoints, $\sum u_n$ converges (absolutely) by the comparison test with $\sum n^{-3}$. Hence, the interval of convergence of the given series is $\left[\frac{5}{3}, \frac{7}{3}\right]$.

13. $\displaystyle\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n.$