1. What is the equation of the tangent line to the graph of

$$y = \arctan(\sqrt{2x+1})$$

at the point where $x = 0$?

2. Evaluate the integrals.

a. $\displaystyle\int_0^5 x\sqrt{9-x}\,dx$ b. $\displaystyle\int \sec^6(t)\sqrt{\tan(t)}\,dt$ c. $\displaystyle\int_0^{\frac{1}{10}} \arcsin(5x)\,dx$

d. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ e. $\displaystyle\int \frac{3x^3+3x^2-12x+20}{x^4+4x^2}\,dx$

f. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ g. $\displaystyle\int e^{2x}\cos(x)\,dx$

3. Evaluate the improper integrals.

a. $\displaystyle\int_0^\infty \frac{\arctan(x)}{1+x^2}\,dx$ b. $\displaystyle\int_0^1 x\log(x)\,dx$

4. Evaluate the limits.

a. $\displaystyle\lim_{x\to\infty}\left(1+\frac{4}{x-2}\right)^x$ b. $\displaystyle\lim_{x\to 1}\left\{\frac{1}{x-1}-\frac{1}{\log(x)}\right\}$ c. $\displaystyle\lim_{x\to 0^+} e^{-1/x}\log(x)$

5. Let f be a function with a continuous derivative on $[0,\infty)$ such that $f(0) = 2$ and $0 \leqslant f(x) \leqslant 100$ for $x \geqslant 0$.

$$\text{If} \quad \int_0^\infty e^{-x} f(x)\,dx = 15, \quad \text{then evaluate} \quad \int_0^\infty e^{-x} f'(x)\,dx.$$

6. Let $\mathscr{R}$ be the region bounded by the graphs of $y = x$, $y = e^x$, $x = 0$, $x = 4$.

a. Compute the area of $\mathscr{R}$.

b. Write an integral which is equal to the volume of the solid generated by rotating $\mathscr{R}$ about: i. the x-axis; ii. the line defined by $x = 4$.

c. Find the volume of the solid generated by rotating $\mathscr{R}$ about the y-axis.

7. Solve the differential equation

$$y\frac{dy}{dx}\sqrt{x^2+1} = x; \quad y(0) = 1.$$

8. Does the sequence $\left\{\sqrt{n^4+n^3}-n^2\right\}$ converge? If so, find its limit. Justify your answer.

9. Mark each statement as true or false. Justify your answers.

a. If $a_n \geqslant 0$ for $n \geqslant 1$ and $\sum a_n$ converges then $\sum \sin(a_n)$ converges.

b. If $a_n \geqslant 0$ for $n \geqslant 1$ and $\sum \sin(a_n)$ converges then $\sum a_n$ converges.

c. If $0 \leqslant a_n < \frac{1}{2}\pi$ for $n \geqslant 1$ and $\sum \sin(a_n)$ converges, then $\sum a_n$ converges.

10. Determine whether each of the following series converges or diverges. State the tests you use, and verify that the conditions for using them are satisfied.

a. $\displaystyle\sum_{n=0}^\infty \frac{\sec^2(n)}{\sqrt{n+1}}$ b. $\displaystyle\sum_{n=1}^\infty \frac{n^{3n}}{(3n)!}$ c. $\displaystyle\sum_{n=0}^\infty \frac{\sqrt[3]{n^4+3}}{3n^2+7}$

11. Label each series as absolutely convergent, conditionally convergent or divergent. Justify all answers with precise references to convergence tests (and verify the hyptheses of each convergence test used).

a. $\displaystyle\sum_{n=2}^\infty (-1)^n \frac{\log(n)}{n}$ b. $\displaystyle\sum_{k=0}^\infty (-1)^k e^{-\sqrt{k}}$

12. If a power series

$$\sum_{n=0}^\infty a_n(x-3)^n$$

converges at $x = 5$, must it also converge at $x = 0$? Must it also converge at $x = 4$? Justify your answers.

13. Find the radius and interval of convergence of the power series

$$\sum_{k=1}^\infty \frac{(-1)^k \log(1+\sqrt{k^{-1}})(2x-1)^k}{3^k\sqrt[3]{k}}.$$

14. Find the Taylor series of $f(x) = x\log(x)$ centred at $x = 1$. For which values of x does the series converge?

Solutions

1. If $y = \arctan\sqrt{2x+1}$, then

$$\frac{dy}{dx} = \frac{1}{1+(2x+1)} \cdot \frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)\sqrt{2x+1}},$$

hence $\frac{dy}{dx} = \frac{1}{2}$ if $x = 0$. Now $y = \arctan(1) = \frac{1}{4}\pi$ if $x = 0$, so an equation of the tangent line in question is $x - 2y = -\frac{1}{2}\pi$.

2. a. Partial integration yields

$$\int_0^5 x\sqrt{9-x}\,dx = -\left\{\tfrac{2}{3}x(9-x)^{3/2} + \tfrac{4}{15}(9-x)^{5/2}\right\}\Big|_0^5 = -\tfrac{80}{3} + \tfrac{4}{15}\cdot 211 = \tfrac{148}{5}.$$

b. Let $y = \tan(t)$, so that $dy = \sec^2(t)\,dt$, and $\sec^4(t) = (y^2+1)^2 = y^4 + 2y^2 + 1$. Therefore,

$$\int \sec^6(t)\sqrt{\tan(t)}\,dt = \int \left\{y^{9/2} + 2y^{5/2} + y^{1/2}\right\} dy$$
$$= \tfrac{2}{11}\tan^{11/2}(t) + \tfrac{4}{7}\tan^{7/2}(t) + \tfrac{2}{3}\tan^{3/2}(t) + \beta.$$

c. If $\vartheta = \arcsin(5x)$ then $5x = \sin(\vartheta)$, $5\,dx = \cos(\vartheta)\,d\vartheta$, $\vartheta = 0$ if $x = 0$ and $\vartheta = \frac{1}{6}\pi$ if $x = \frac{1}{10}$. Partial integration then gives

$$\int_9^{\frac{1}{10}} \arcsin(5x)\,dx = \tfrac{1}{5}\int_0^{\frac{1}{6}\pi} \vartheta\cos(\vartheta)\,d\vartheta = \tfrac{1}{5}\left\{\vartheta\sin(\vartheta) + \cos(\vartheta)\right\}\Big|_0^{\frac{1}{6}\pi}$$
$$= \tfrac{1}{60}\pi + \tfrac{1}{10}\sqrt{3} - \tfrac{1}{5}.$$

d. By inspection,

$$\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}} = \arcsin(e^x) + \delta.$$

e. The resolution into partial fractions of the integrand has the form

$$\frac{3x^3 + 3x^2 - 12x + 20}{x^2(x^2+4)} = \frac{a}{x} + \frac{5}{x^2} + \frac{bx+c}{x^2+4},$$

where the second coefficient is found by inspection (covering). Clearing denominators gives

$$3x^3 + 3x^2 - 12x + 20 = (ax+5)(x^2+4) + x^2(bx+c),$$

and then comparing the cubic, quadratic and linear coefficients gives $a+b = 3$, $5+c = 3$, or $c = -2$, and $4a = -12$, or $a = -3$. Then $a+b = 3$ gives $b = 6$. Thus, the integral is equal to

$$\int \left\{-\frac{3}{x} + \frac{5}{x^2} + \frac{6x-2}{x^2+4}\right\} dx = -\frac{5}{x} + 3\log\frac{x^2+4}{|x|} - \arctan\left(\tfrac{1}{2}x\right) + \varepsilon.$$

f. Since (by the product rule for differentiation)

$$\frac{d}{dx}\left\{x\sqrt{4-x^2}\right\} = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{4}{\sqrt{4-x^2}} - \frac{2x^2}{\sqrt{4-x^2}},$$

it follows that

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = -\tfrac{1}{2}x\sqrt{4-x^2} + 2\arcsin\left(\tfrac{1}{2}x\right) + \varphi.$$

g. By the product rule for differentiation,

$$\frac{d}{dx}\left\{e^{2x}\cos(x)\right\} = 2e^{2x}\cos(x) - e^{2x}\sin(x), \quad \text{and}$$
$$\frac{d}{dx}\left\{e^{2x}\sin(x)\right\} = e^{2x}\cos(x) + 2e^{2x}\sin(x).$$

Adding twice the first equation to the second and then rearranging the corresponding integral equation, gives

$$\int e^{2x}\cos(x)\,dx = \tfrac{1}{5}e^{2x}\left(2\cos(x) + \sin(x)\right) + \gamma.$$

3. a. Integrating by inspection gives

$$\int_0^\infty \frac{\arctan(x)}{1+x^2}\,dx = \lim_{t\to\infty} \tfrac{1}{2}(\arctan t)^2\Big|_0^t = \tfrac{1}{8}\pi^2.$$

b. Partial integration gives

$$\int x\log(x)\,dx = \tfrac{1}{2}x^2\log(x) - \tfrac{1}{2}\int x\,dx = \tfrac{1}{4}x^2(2\log(x) - 1),$$

so by elementary properties of the logarithm, it follows that

$$\int_0^1 x\log(x)\,dx = -\tfrac{1}{4} - \tfrac{1}{4}\lim_{t\to 0^+} t^2(2\log(t) - 1) = -\tfrac{1}{4}.$$

(The definition of the logarithm immediately gives $1 - 1/t < \log(t) < 0$, and hence $-t < t^2\log(t) < 0$, for $0 < t < 1$.)

4. a. Since $\lim_{t\to 0}(1+t)^{1/t} = e$, it follows that

$$\lim_{x\to\infty}\left(1 + \frac{4}{x-2}\right)^x = \lim_{x\to\infty}\left(1 + \frac{4}{x-2}\right)^{\frac{x-2}{4}\cdot\frac{4x}{x-2}} = e^4.$$

b. Upon combining terms and expanding $\log(x)$ about 1 gives

$$\lim_{x\to 1}\left\{\frac{1}{x-1} - \frac{1}{\log(x)}\right\} = \lim_{x\to 1} = \frac{-\frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots}{(x-1)^2 - \frac{1}{2}(x-1)^3 + \cdots} = -\tfrac{1}{2}.$$

c. Since the exponential function dominates the logarithm, it follows that

$$\lim_{x\to 0^+} e^{-1/x}\log(x) = 0.$$

(If $y > 1$ then $0 < \log(y) < \sqrt{y}$ and $y < e^y$, so $-\sqrt{x}^{-1} < \log(x) < 0$ and $e^{-1/x} < x$, and hence $-\sqrt{x} < e^{-1/x}\log(x) < 0$, provided $0 < x < 1$.)

5. Partial integration, integrating the derivative of f, gives

$$\int_0^\infty e^{-x}f'(x)\,dx = \lim_{x\to\infty} e^{-x}f(x) - 2 + \int_0^\infty e^{-x}f(x)\,dx = -2 + 15 = 13,$$

since $0 \leqslant e^{-x}f(x) \leqslant 100e^{-x}$ if $x \geqslant 0$, so that $\lim_{x\to\infty} e^{-x}f(x) = 0$.

6. a. The area of $\mathscr{R}$ is equal to

$$\int_0^4 (e^x - x)\,dx = \left\{e^x - \tfrac{1}{2}x^2\right\}\Big|_0^4 = e^4 - 9.$$

b. The solid generated by revolving $\mathscr{R}$ about the x-axis consists of annuli of inner radius x and outer radius e^x, for $0 \leqslant x \leqslant 4$, so its volume is equal to

$$\pi\int_0^4 (e^{2x} - x^2)\,dx = \tfrac{1}{6}\pi(3e^{2x} - 2x^3)\Big|_0^4 = \tfrac{1}{6}\pi(3e^8 - 131).$$

The solid generated by revolving $\mathscr{R}$ about the line defined by $x = 4$ consists of concentric cylindrical shells of radius $4-x$ and height $e^x - x$, for $0 \leqslant x \leqslant 4$, so its volume is equal to

$$2\pi\int_0^4 (4-x)(e^x - x)\,dx = 2\pi\left\{e^x(5-x) + \tfrac{1}{3}x^3 - 2x^2\right\}\Big|_0^4 = \tfrac{2}{3}\pi(3e^4 - 47).$$

c. The solid obtained by revolving $\mathscr{R}$ about the y-axis consists of concentric cylindrical shells of radius x and height $e^x - x$, for $0 \leqslant x \leqslant 4$, so its volume is equal to

$$2\pi\int_0^4 x(e^x - x)\,dx = \tfrac{2}{3}\pi\left\{3e^x(x-1) - x^3\right\}\Big|_0^4 = \tfrac{2}{3}\pi(9e^4 - 61).$$

7. Separating the variables and integrating yields

$$\int y\,dy = \int \frac{x}{\sqrt{x^2+1}}\,dx, \quad \text{or} \quad y^2 = 2\sqrt{x^2+1} + C.$$

Since $y = 1$ if $x = 0$, it follows that $C = -1$. Therefore, $y = \sqrt{2\sqrt{x^2+1} - 1}$.

8. The sequence diverges (to ∞), since (by rationalizing the numerator)

$$\lim_{n\to\infty}\left\{\sqrt{n^4+n^3} - n^2\right\} = \lim_{n\to\infty} \frac{n}{1+\sqrt{1+1/n}} = \infty.$$

9. Recall that if $0 < \vartheta < \frac{1}{3}\pi$ then $\frac{1}{2}\vartheta < \sin(\vartheta) < \vartheta$.
a. Since $\sum a_n$ converges, $\lim a_n = 0$, so there is a positive integer k such that $0 \leqslant a_n < \frac{1}{2}\pi$, and hence $0 \leqslant \sin(a_n) \leqslant a_n$, if $n \geqslant k$. So the comparison test implies that $\sum \sin(a_n)$ converges with $\sum a_n$. Therefore, the statement is true.
b. If $a_n = \pi n$ then $\sum a_n$ diverges by the vanishing condition, but $\sum \sin(a_n)$ converges (to 0), since each of its terms is zero. So the statement is false.
c. If $\sum \sin(a_n)$ is convergent and $0 \leqslant a_n < \frac{1}{2}\pi$, then $\lim \sin(a_n) = 0$ and $a_n = \arcsin(\sin(a_n))$, so $\lim a_n$. Hence, there is a positive integer k such that $0 \leqslant a_n < \frac{1}{3}\pi$, and hence $0 \leqslant a_n \leqslant 2\sin(a_n)$, provided $n \geqslant k$. The comparison test implies that $\sum a_n$ converges with $\sum \sin(a_n)$. So the statement is true.

10. a. Since $\alpha_{n-1} = \sec^2(n-1)/\sqrt{n} \geqslant n^{-1/2}$ for $n \geqslant 1$, the comparison test implies that the series $\sum \alpha_n$ diverges with the p-series $\sum n^{-1/2}$ ($p = \frac{1}{2} \leqslant 1$).
b. If $\alpha_n = n_{3n}/(3n)!$, then $\alpha_n > 0$ for $n \geqslant 1$ and

$$\lim \frac{\alpha_{n+1}}{\alpha_n} = \lim\left\{\frac{(n+1)^2}{3(3n+2)(3n+1)} \cdot \left(1+\frac{1}{n}\right)^{3n}\right\} = (e/3)^3,$$

which is smaller than one. Therefore, the ratio test implies that the series $\sum \alpha_n$ is convergent.
c. If $n \geqslant 1$ then

$$\frac{\sqrt[3]{n^4+3}}{3n^2+7} \geqslant \frac{\sqrt[3]{n^4}}{10n^2} = \frac{1}{10}n^{-2/3},$$

so the comparison test implies that the series in question diverges with the p-series $\sum n^{-2/3}$ ($p = \frac{2}{3} \leqslant 1$).

11. a. Since $0 < 1/n < \log(n)/n$ if $n \geqslant 3$, the comparison test implies that the series $\sum \log(n)/n$ diverges with the harmonic series. On the other hand, $\lim \log(n)/n = 0$ by elementary properties of the logarithm, and

$$\frac{d}{dx}\frac{\log(x)}{x} = \frac{1-\log(x)}{x^2} < 0$$

if $x > e$, so $\log(n+1)/(n+1) < \log(n)/n$ if $n \geqslant 3$. Hence, the alternating series test implies that $\sum(-1)^n \log(n)/n$ is convergent. Therefore, $\sum(-1)^n\log(n)/n$ is conditionally convergent.

b. Since $\lim k^2 e^{-\sqrt{k}} = 0$ by elementary proerties of the exponential function, the limit comparison test implies that the series $\sum e^{-\sqrt{k}}$ converges with the p-series $\sum k^{-2}$ ($p = 2 > 1$). Therefore, $\sum(-1)^k e^{-\sqrt{k}}$ is absolutely convergent.

12. Since the given power series is centred at 3 and convergent at 5, its radius of convergence is not less than two, but might be no larger than two. Therefore, the series certainly converges at 4, but need not converge at 0.

13. If $x \neq \frac{1}{2}$ and

$$\alpha_k = \frac{(-1)^k \log(1+\sqrt{k^{-1}})(2x-1)^k}{3^k\sqrt[3]{k}},$$

then

$$\lim\left|\frac{\alpha_{k+1}}{\alpha_k}\right| = \tfrac{1}{3}|2x-1|\lim\left\{\frac{\sqrt[3]{k}}{\sqrt[3]{(k+1)}} \cdot \frac{\log(1+\sqrt{(k+1)^{-1}})}{\log(1+\sqrt{k^{-1}})}\right\} = \tfrac{1}{3}|2x-1|.$$

So the ratio test implies that $\sum \alpha_k$ is absolutely convergent if $|2x-1| < 3$, *i.e.*, $-1 < x < 2$, and is divergent $x < -1$ or else $x > 2$. Let $a_k = k^{-1/3}\log(1+\sqrt{k^{-1}})$; if $x = -1$ then $\alpha_k = a_k$, and if $x = 2$ then $\alpha_k = (-1)^k a_k$. Recall that if $x > -1$ and $x \neq 0$ then $x/(1+x) < \log(1+x) < x$; so in particular

$$0 < \frac{\sqrt{k^{-1}}}{1+\sqrt{k^{-1}}} < \log(1+\sqrt{k^{-1}}),$$

and thus

$$a_k > \frac{k^{-5/6}}{1+\sqrt{k^{-1}}} \geqslant \tfrac{1}{2}k^{-5/6} > 0 \qquad \text{if} \quad k \geqslant 1.$$

Hence, the comparison test implies that the series $\sum a_k$ diverges with the p-series $\sum k^{-5/6}$ ($p = \frac{5}{6} \leqslant 1$). On the other hand, $0 < \sqrt[3]{(k+1)^{-1}} < \sqrt[3]{k^{-1}}$ and $0 < \log(1+\sqrt{(k+1)^{-1}}) < \log(1+\sqrt{k^{-1}})$, and hence $a_{k+1} < a_k$ if $k \geqslant 1$, and $\lim \sqrt[3]{k^{-1}} = \lim \log(1+\sqrt{k^{-1}}) = 0$, so $\lim a_k = 0$. So the alternating series test implies that $\sum(-1)^k a_k$ is convergent. Therefore, the radius of convergence of $\sum \alpha_k$ is $\frac{3}{2}$, and its interval of convergence is $(-1, 2]$.

14. Using the Maclaurin series of $\log(1+t)$, where $t = x-1$, gives

$$\begin{aligned} x\log(x) &= (1+(x-1))\log(1+(x-1)) \\ &= (1+(x-1))\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k \\ &= (x-1) + \sum_{k=2}^{\infty}\frac{(-1)^k}{k(k-1)}(x-1)^k, \end{aligned}$$

at least if $-1 < x-1 \leqslant 1$, or $0 < x \leqslant 2$. If $x = 0$ then the series is

$$-1 + \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = -1 + \sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) = -1 + \lim_{k\to\infty}\left(1 - \frac{1}{k}\right) = 0.$$

Therefore, the interval of convergence of the Taylor series is $[0, 2]$.