1. Find the slope of the line tangent to the graph of

$$y = \frac{5}{5 + \frac{16}{5}x^2} + 6\arctan \tfrac{4}{5}x$$

at the point where $x = \frac{5}{4}$.

2. Evaluate each of the following integrals.

a. $\displaystyle\int (xe^{-x^2} + \pi)\,dx$ b. $\displaystyle\int \frac{\log(\log x)}{x}\,dx$

c. $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$ d. $\displaystyle\int_0^1 x\arctan x\,dx$

e. $\displaystyle\int \frac{2x^2-3x+4}{(x-1)(x^2+2)}\,dx$ f. $\displaystyle\int \frac{\sec^4(\sin x)\tan^2(\sin x)}{\sec x}\,dx$

3. Evaluate the improper integrals.

a. $\displaystyle\int_{-10}^{\infty} e^{-x}\,dx$ b. $\displaystyle\int_{-2}^{0} \frac{dx}{(x+1)^3}$

4. Evaluate the limits.

a. $\displaystyle\lim_{x\to\infty} \left(\frac{1}{x}\right)^x$ b. $\displaystyle\lim_{x\to 0} \frac{\arcsin(11x)}{\arcsin(5x)}$

5. Compute the area of the region bounded by the curves $y = x^3 - 2x^2 + x + 1$ and $y = -x^2 + x + 1$.

6. Let $\mathscr{R}$ be the region between the graphs of

$$y = \frac{\cos x}{x}, \quad y = \frac{1}{x^2}, \quad x = \tfrac{1}{4}\pi \quad \text{and} \quad x = \tfrac{1}{2}\pi.$$

a. Set up, but do not evaluate, an integral that represents the volume of the solid generated by revolving $\mathscr{R}$ about the x-axis.

b. Find the volume of the solid generated by revolving $\mathscr{R}$ about the y-axis.

7. Solve the differential equation

$$(x+1)y' = -y^2; \quad y(0) = \tfrac{1}{2}.$$

8. Determine whether the sequence converges or diverges. If the sequence converges, find its limit. Justify your answers.

a. $\displaystyle\left\{\frac{(n+2)!}{n!}\right\}$ b. $\displaystyle\left\{\csc\tfrac{1}{n} - \cot\tfrac{1}{n}\right\}$ c. $\displaystyle\left\{\left(\frac{n}{1+n}\right)^n\right\}$

9. Find the exact sum of the series.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{3^n}$ b. $\displaystyle\sum_{n=2}^{\infty} \left[\frac{1}{\log(n+1)} - \frac{1}{\log(n)}\right]$

10. Determine whether each of the following series converges or diverges. State the tests you use, and verify that the conditions for using them are satisfied.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2 2^n}{(2n+1)!}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos n}{5^n}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$ d. $\displaystyle\sum_{n=1}^{\infty} \frac{6+\log(n)}{\sqrt{n}}$

11. Label each series as absolutely convergent, conditionally convergent or divergent. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\arctan n}\right)^n$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n+1)}}$

12. Determine the radius and interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\log(n)}(x-3)^n.$$

13. Find the Taylor series centred at 5 of

$$f(x) = \frac{1}{2-x},$$

and state its interval of convergence.

Solution outlines

1. Let $t = \frac{4}{5}x$; so $dt/dx = \frac{4}{5}$ for all x, and therefore

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=\frac{5}{4}} &= \left.\frac{dy}{dt}\right|_{t=1} \cdot \left.\frac{dt}{dx}\right|_{x=\frac{5}{4}} \\ &= \left.\frac{d}{dt}\left\{\frac{1}{t^2+1} + 6\arctan t\right\}\right|_{t=1} \cdot \frac{4}{5} \\ &= \frac{8}{5}\cdot \left.\frac{3t^2-t+3}{(t^2+1)^2}\right|_{t=1} \\ &= 2.\end{aligned}$$

2. a. $\displaystyle\int (\pi + xe^{-x^2})\,dx = \pi x - \tfrac{1}{2}e^{-x^2} + C$ by inspection.

b. Integrate by parts after changing the variable of integration to $t = \log(x)$.

$$\begin{aligned}\int \frac{\log(\log x)}{x}\,dx &= \int \log(t)\,dt \\ &= t\log(t) - \int dt \\ &= (\log(x))(\log(\log x) - 1) + C\end{aligned}$$

c. Integrate the fractional power of $9 - x^2$ twice, first with partial integration.

$$\begin{aligned}\int \frac{x^3}{\sqrt{9-x^2}}\,dx &= -x^2\sqrt{9-x^2} + 2\int x\sqrt{9-x^2}\,dx \\ &= -x^2\sqrt{9-x^2} - \tfrac{2}{3}(9-x^2)^{3/2} + C \\ &= -\tfrac{1}{3}(x^2+18)\sqrt{9-x^2} + C\end{aligned}$$

d. Partial integration yields (taking the primitive $\frac{1}{2}(x^2+1)$ of x),

$$\begin{aligned}\int_0^1 x\arctan x\,dx &= \left.\tfrac{1}{2}(x^2+1)\arctan x\right|_0^1 - \tfrac{1}{2}\int_0^1 \frac{x^2+1}{x^2+1}\,dx \\ &= \tfrac{1}{4}\pi - \left.\left(\tfrac{1}{2}x\right)\right|_0^1 \\ &= \tfrac{1}{4}\pi - \tfrac{1}{2}.\end{aligned}$$

e. Resolve the integrand into partial fractions and then split the second fraction.

$$\begin{aligned}\int \frac{2x^2-3x+4}{(x-1)(x^2+2)}\,dx &= \int\left\{\frac{1}{x-1} + \frac{x-2}{x^2+2}\right\}dx \\ &= \int\left\{\frac{1}{x-1} + \frac{x}{x^2+2} - \frac{2}{x^2+2}\right\}dx \\ &= \log|x-1| + \tfrac{1}{2}\log(x^2+2) - \sqrt{2}\arctan\tfrac{1}{2}x\sqrt{2} + C \\ &= \log\left\{|x-1|\sqrt{x^2+2}\right\} - \sqrt{2}\arctan\tfrac{1}{2}x\sqrt{2} + C\end{aligned}$$

f. Changing the variable of integration to $t = \tan(\sin x)$ gives

$$\begin{aligned}\int \frac{\sec^4(\sin x)\tan^2(\sin x)}{\sec x}\,dx &= \int t^2(t^2+1)\,dt \\ &= \tfrac{1}{5}t^5 + \tfrac{1}{3}t^3 + C \\ &= \tfrac{1}{15}\big(3\tan^2(\sin x) + 5\big)\tan^3(\sin x) + C.\end{aligned}$$

3. a. Integrating by inspection gives

$$\int_{-10}^{\infty} e^{-x}\,dx = \lim_{t\to\infty}\int_{-10}^{t} e^{-x}\,dx = \lim_{t\to\infty}(-e^{-t}) + e^{10} = e^{10}.$$

b. Since

$$\lim_{\alpha\to -1^-}\int_{-2}^{\alpha}\frac{dx}{(x+1)^3} = \lim_{\alpha\to -1^-}\frac{-1}{2(\alpha+1)^2}+\frac{1}{2} = -\infty,$$

it follows that the improper integral

$$\int_{-2}^{0}\frac{dx}{(x+1)^3}$$

diverges.

4. a. Writing the expression base e gives

$$\lim_{x\to\infty}\left(\frac{1}{x}\right)^{x} = \lim_{x\to\infty} e^{-x\log(x)} = 0.$$

b. One application of l'Hôpital's rule gives

$$\lim_{x\to 0}\frac{\arcsin(11x)}{\arcsin(5x)} = \lim_{x\to 0}\frac{11\sqrt{1-(5x)^2}}{5\sqrt{1-(11x)^2}} = \tfrac{11}{5}.$$

5. The curves meet where $x^3 - 2x^2 + x + 1 = -x^2 + x + 1$, or equivalently $0 = x^3 - x^2 = x^2(x-1)$; *i.e.*, where $x = 0$ or $x = 1$. If $0 < x < 1$ then $x^2(x-1) < 0$, so the cubic lies below the parabola, and the area of the region they enclose is equal to

$$\int_0^1 (-x^3 + x^2)\,dx = \left(-\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3\right)\Big|_0^1 = \tfrac{1}{12}.$$

6. a. The solid in question consists of annular cross sections of inner radius $(\cos x)/x$ and outer radius $1/x^2$, for $\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{2}\pi$, so its volume is equal to

$$\pi\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\left\{\frac{1}{x^4} - \frac{\cos^2 x}{x^2}\right\}dx.$$

b. The solid in question can be decomposed into cylindrical shells of radius x and height $1/x^2 - (\cos x)/x$, for $\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{2}\pi$, so its volume is equal to

$$2\pi\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} x\left(\frac{1}{x^2} - \frac{\cos x}{x}\right)dx = 2\pi(\log(x) - \sin(x))\Big|_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = \pi(2\log(2) + \sqrt{2} - 2).$$

7. Separating variables and integrating gives

$$-\int\frac{dy}{y} = \int\frac{dx}{x+1}, \quad\text{and so}\quad \frac{1}{y} = \log(x+1) + 2,$$

since $y(0) = \frac{1}{2}$. Therefore

$$y = \frac{1}{2 + \log(x+1)}$$

(the solution being valid for $x > -1$).

8. a.

$$\lim_{n\to\infty}\frac{(n+2)!}{n!} = \lim_{n\to\infty}\{(n+2)(n+1)\} = \infty$$

b. If $x = 1/n$ then the limit of the sequence is equal to

$$\lim_{x\to 0^+}\frac{1-\cos(x)}{\sin(x)} = \lim_{x\to 0^+}\frac{\sin(x)}{1+\cos(x)} = 0.$$

c. Revising the expression in the limit gives

$$\lim_{n\to\infty}\left(\frac{n}{1+n}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = e^{-1}.$$

9. a. This series is a geometric series with first term 4 and common ratio $-\frac{2}{3}$, so

$$\sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+2}}{3^n} = \frac{4}{1+2/3} = \tfrac{12}{5}.$$

b. Expanding and simplifying the partial sums gives

$$\sum_{n=2}^{\infty}\left\{\frac{1}{\log(n+1)} - \frac{1}{\log(n)}\right\} = \lim_{n\to\infty}\left\{\frac{1}{\log(n+2)} - \frac{1}{\log(2)}\right\} = -\frac{1}{\log(2)}.$$

10. a. Let $\alpha_n = (n!)^2 2^n/(2n+1)!$; then

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\frac{n+1}{2n+3} = \tfrac{1}{2},$$

which is smaller than one. Therefore, $\sum\alpha_n$ converges by the ratio test.

b. Since

$$0 < \frac{2+\cos n}{5^n} < \frac{3}{5^n}$$

for $n \geqslant 1$, and $\sum\left(\frac{1}{5}\right)^n$ is a convergent geometric series,

$$\sum_{n=1}^{\infty}\frac{2+\cos n}{5^n}$$

converges by the comparison test.

c. Since

$$0 < \frac{\arctan n}{n^2+1} < \frac{2}{\pi n^2}$$

for $n \geqslant 1$, and $\sum n^{-2}$ is a convergent p-series,

$$\sum_{n=1}^{\infty}\frac{\arctan n}{n^2+1}$$

converges by the comparison test.

d. Since

$$\frac{6+\log(n)}{\sqrt{n}} \geqslant \frac{6}{\sqrt{n}}$$

for $n \geqslant 1$, and $\sum n^{-1/2}$ is a divergent p-series,

$$\sum_{n=1}^{\infty}\frac{6+\log(n)}{\sqrt{n}}$$

diverges by the comparison test.

11. a. Let $\alpha_n = (-1)^n/(\arctan n)^n$; then

$$\lim_{n\to\infty}\sqrt[n]{|\alpha_n|} = \lim_{n\to\infty}(\arctan n)^{-1} = 2/\pi,$$

which is smaller than one. Therefore, $\sum\alpha_n$ is absolutely convergent by the root test.

b. Let $\alpha_n = \big(n(n+1)\big)^{-1/2}$. Since $n(n+1)$ is positive and increasing, α_n is positive and decreasing, for $n \geqslant 1$. Also, $\lim_{n\to\infty}\alpha_n = 0$, so $\sum(-1)^n\alpha_n$ converges by the alternating series test. However,

$$\lim_{n\to\infty}\frac{\alpha_n}{n^{-1}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+1/n}} = 1,$$

so $\sum\alpha_n$ diverges with the harmonic series by the limit comparison test. Therefore, $\sum(-1)^n\alpha_n$ is conditionally convergent.

12. If

$$\alpha_n = \frac{(-1)^n}{n\log(n)}(x-3)^n;$$

and $x \neq 3$, then

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\left\{\frac{n}{n+1}\cdot\frac{\log(n)}{\log(n+1)}|x-3|\right\} = |x-3|,$$

so the ratio test implies that $\sum\alpha_n$ converges if $2 < x < 4$ and diverges if $x < 2$ or $x > 4$. If $x = 2$ then

$$\sum_{n=2}^{\infty}\alpha_n = \sum_{n=2}^{\infty}\frac{1}{n\log(n)}$$

is a divergent logarithmic p-series, and if $x = 4$ then

$$\sum_{n=2}^{\infty}\alpha_n = \sum_{n=2}^{\infty}\frac{(-1)^n}{n\log(n)}$$

converges by the alternating series test, since $\{1/(n\log(n))\}$ is positive, decreasing and converges to zero. Therefore, the interval of convergence of $\sum\alpha_n$ is $(2, 4]$.

13. The Taylor series in question is a geometric series, as follows:

$$\frac{1}{2-x} = -\frac{1}{3}\cdot\frac{1}{1-\left(-\frac{1}{3}(x-5)\right)} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{3^{n+1}}(x-5)^n.$$

The interval of convergence of this series is $(2, 8)$, where its ratio $\frac{1}{3}|x-5| < 1$.