1. Evaluate $\dfrac{d}{dx}\left\{\sin\left(\arccos\sqrt{1-x^2}\,\right)\right\}$, and simplify your answer.

2. Evaluate the following integrals.

a. $\displaystyle\int_1^{\sqrt{2}} \frac{4+2\sqrt{x^2-1}}{x\sqrt{x^2-1}}\,dx$ b. $\displaystyle\int_1^5 \frac{x+2}{\sqrt{2x-1}}\,dx$

c. $\displaystyle\int e^{-2x}\cos 6x\,dx$ d. $\displaystyle\int \sqrt{t+1}\log\sqrt{t+1}\,dt$

e. $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^3 x\cos^3 x\,dx$ f. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-36}}$ g. $\displaystyle\int \frac{x+4}{x(x^2+2)}\,dx$

3. Evaluate the following improper integrals.

a. $\displaystyle\int_2^{\infty} \frac{1}{1-x^2}\,dx$ b. $\displaystyle\int_0^2 \frac{x}{x^2-4}\,dx$

4. Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0^+}\frac{(\log x)^2}{1+x^{-1}}$ b. $\displaystyle\lim_{x\to 0}(\sec x)^{\cot^2 x}$ c. $\displaystyle\lim_{x\to\infty}\left\{\frac{x^2+2}{x-3}-\frac{(x-2)^3}{x^2+1}\right\}$

5. Find the area of the region (in quadrant I) bounded by the graphs of

$$y=\frac{2}{x},\quad y=\frac{3x}{x^2+2}\quad\text{and}\quad x=1.$$

Give the exact answer in simplified form only: no decimals.

6. Let $\mathscr{R}$ be the region bounded by the graphs of

$$y=\frac{x^2}{4},\quad y=x^3-3x+3,\quad x=-2\quad\text{and}\quad x=2.$$

a. Set up, *but do not evaluate*, an integral that represents the volume of the solid generated by revolving $\mathscr{R}$ about the x-axis.

b. Find the volume of the solid generated by revolving $\mathscr{R}$ about the line $x=3$. *Give the exact answer in simplified form only: no decimals.*

7. Give the explicit solution, in simplified form, of the initial value problem

$$\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{1+x^2};\quad y(1)=0.$$

8. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series whose n^{th} partial sum is given by $s_n=\dfrac{2n+1}{n+2}$.

a. Evaluate $\displaystyle\sum_{n=1}^{\infty} a_n$. b. Find a_5.

9. What can you say about the convergence of each series based only on the limit of its general term?

a. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n}$ b. $\frac{1}{2}+1+\frac{1}{4}+1+\frac{1}{8}+1+\frac{1}{16}+1+\cdots$

10. Determine whether each of the following series converges or diverges; if it converges, find the sum. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{5(-4)^{n+2}}{3^{2n+1}}$ b. $\displaystyle\sum_{n=1}^{\infty}\log\frac{2n-1}{2n+1}$

11. Determine whether each of the following series converges or diverges. State the tests you use and verify that the conditions for using them are satisfied.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ b. $\displaystyle\sum_{k=1}^{\infty}\frac{\cos^2(k)}{k\sqrt{k}}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{e^{\sqrt{n}}}{\sqrt{n}}$ d. $\displaystyle\sum_{n=2}^{\infty}\sin\left(\frac{2}{n}\right)$

12. Determine whether each of the following series converges absolutely, conditionally or diverges. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-n}{2n+1}\right)^{3n}$ b. $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\log n}{\sqrt{n}}$

13. Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty}\frac{3^n}{2n+1}(x-2)^{n+1}.$$

14. Find the Taylor series of $f(x)=\cos(2x)$ centred at $\frac{1}{2}\pi$. State the first four *non-zero* terms and give the formula for the n^{th} term.

Solution outlines

1. $\sin\left(\arccos\sqrt{1-x^2}\,\right)=|x|$, and hence $\dfrac{d}{dx}\left\{\sin\left(\arccos\sqrt{1-x^2}\,\right)\right\}=\dfrac{x}{|x|}$ (1 if $x>0$ and -1 if $x<0$).

2. a. Separating terms and simplifying reveals two basic integrals:

$$\int_1^{\sqrt{2}}\frac{4+2\sqrt{x^2-1}}{x\sqrt{x^2-1}}\,dx=(4\,\text{arcsec}\,x+2\log x)\Big|_1^{\sqrt{2}}=\pi+\log 2.$$

b. Partial integration gives

$$\int_1^5\frac{x+2}{\sqrt{2x-1}}\,dx=\left\{(x+2)\sqrt{2x-1}-\tfrac{1}{3}(2x-1)^{3/2}\right\}\Big|_1^5$$
$$=\tfrac{1}{3}(x+7)\sqrt{2x-1}\Big|_1^5=\tfrac{28}{3}.$$

c. By the product rule for differentiation,

$$\frac{d}{dx}\left\{e^{-2x}\cos 6x\right\}=-2e^{-2x}\cos 6x-6e^{-2x}\sin x,\quad\text{and}$$
$$\frac{d}{dx}\left\{e^{-2x}\sin 6x\right\}=\quad 6e^{-2x}\cos 6x-2e^{-2x}\sin x.$$

Subtracting the first equation from three times the second, and then rearranging the corresponding integral equation, one obtains

$$\int e^{-2x}\cos 6x\,dx=\tfrac{1}{20}e^{-2x}(3\sin 6x-\cos 6x)+C.$$

d. Integrate by parts after revising the logarithmic factor, to obtain

$$\tfrac{1}{2}\int\sqrt{t+1}\log(t+1)\,dt=\tfrac{1}{3}(t+1)^{3/2}\log(t+1)-\tfrac{2}{9}(t+1)^{3/2}+C$$
$$=\tfrac{1}{9}\left(3\log(t+1)-2\right)\sqrt{(t+1)^3}+C.$$

e. Letting $t=\sin x$ gives

$$\int_0^{\frac{1}{2}\pi}\sin^3 x\cos^3 x\,dx=\int_0^1 t^3(1-t^2)\,dt=\tfrac{1}{12}t^4(3-2t^2)\Big|_0^1=\tfrac{1}{12}.$$

f. Let $t=\sqrt{x^2-36}/x$, or $t^2=1-36/x^2$, so that $\frac{1}{36}t\,dt=dx/x^3$, and hence

$$\int\frac{dx}{x^2\sqrt{x^2-36}}=\int\frac{x}{\sqrt{x^2-36}}\cdot\frac{dx}{x^3}=\tfrac{1}{36}\int dt=\frac{\sqrt{x^2-36}}{36x}+C.$$

g. Resolve the integrand into partial fractions and split the last fraction to integrate.

$$\int\frac{x+4}{x(x^2+2)}\,dx=\int\left\{\frac{2}{x}-\frac{2x-1}{x^2+2}\right\}dx$$
$$=\int\left\{\frac{2}{x}-\frac{2x}{x^2+2}+\frac{1}{x^2+2}\right\}dx$$
$$=\log\frac{x^2}{x^2+2}+\tfrac{1}{2}\sqrt{2}\arctan\tfrac{1}{2}x\sqrt{2}+C.$$

3. a.

$$\int_2^{\infty}\frac{dx}{1-x^2}=\lim_{t\to\infty}\int_2^t\frac{dx}{1-x^2}=\lim_{t\to\infty}\tfrac{1}{2}\log\frac{t+1}{t-1}-\tfrac{1}{2}\log 3=-\tfrac{1}{2}\log 3$$

b. The integral diverges, because

$$\int_0^2\frac{x\,dx}{x^2-4}=\lim_{\alpha\to 2^-}\int_0^{\alpha}\frac{x\,dx}{x^2-4}=\lim_{\alpha\to 2^-}\tfrac{1}{2}\log(4-\alpha^2)-\log 2=-\infty.$$

4. a. If $y=1/x$, then

$$\lim_{x\to 0^+}\frac{(\log x)^2}{1+x^{-1}}=\lim_{y\to\infty}\frac{(\log y)^2}{1+y}=0,$$

by elementary properties of the logarithm.

b. Revising the expression in the limit gives

$$\lim_{x\to 0}(1+\sec(x)-1)^{\frac{1}{\sec(x)-1}\cdot\cos(x)\cdot\frac{1-\cos(x)}{\sin^2(x)}} = e^{1/2},$$

since $1-\cos(x) = \frac{1}{2}x^2 - \frac{1}{24}x^4 + \&c.$, $\sin^2(x) = x^2 - \frac{1}{3}x^4 + \&c.$ and $\lim_{t\to 0}(1+t)^{1/t} = e$.

c. Combining terms, and extracting the dominant powers of x from the numerator and denominator, yields

$$\lim_{x\to\infty}\left\{\frac{x^2+2}{x-3} - \frac{(x-2)^3}{x^2+1}\right\} = \lim_{x\to\infty}\frac{9-17/x+44/x^2-24/x^3}{(1-3/x)(1+1/x^2)} = 9.$$

5. Since $2/x - 3x(x^2+2) = (4-x^2)/\big(x(x^2+2)\big)$ is equal to zero if $x=\pm 2$ and is positive if $1 \leqslant x < 2$, the area in question is equal to

$$\int_1^2\left\{\frac{2}{x} - \frac{3x}{x^2+2}\right\}dx = \tfrac{1}{2}\log\frac{x^4}{(x^2+2)^3}\bigg|_1^2 = \tfrac{1}{2}\log 2.$$

6. a. The solid in question can be decomposed into annuli of inner radius $\frac{1}{4}x^2$ and outer radius x^3-3x+3, for $-2 \leqslant x \leqslant 2$, so its volume is equal to

$$\pi\int_{-2}^{2}\left\{(x^3-3x+3)^2 - (\tfrac{1}{4}x^2)^2\right\}dx = \pi\int_0^2\left\{2x^6 - \tfrac{97}{8}x^4 + 18x^2 + 18\right\}dx$$

$$= \pi\left\{\tfrac{2}{7}x^7 - \tfrac{97}{40}x^5 + 6x^3 + 18x\right\}\bigg|_0^2 = \tfrac{1504}{35}\pi.$$

b. The solid in question can be decomposed into cylindrical shells of radius $3-x$ and height $x^3 - \frac{1}{4}x^2 - 3x + 3$, for $-2 \leqslant x \leqslant 2$, so its volume is equal to

$$2\pi\int_{-2}^{2}(3-x)(x^3-\tfrac{1}{4}x^2-3x+3)\,dx = 2\pi\int_0^2\left\{-2x^4 + \tfrac{9}{2}x^2 + 18\right\}dx$$

$$= 2\pi\left\{\tfrac{2}{5}x^5 + \tfrac{3}{2}x^3 + 18x\right\}\bigg|_0^2 = \tfrac{352}{5}\pi.$$

7. Separating variables gives

$$\int\frac{dy}{\sqrt{1-y^2}} = \int\frac{dx}{1+x^2}, \quad\text{or}\quad \arcsin(y) = \arctan(x) - \tfrac{1}{4}\pi,$$

since $y(1) = 0$. Therefore,

$$y = \sin\big(\arctan(x) - \tfrac{1}{4}\pi\big) = \frac{x-1}{\sqrt{2(x^2+1)}}.$$

8. a. The sum of the series is $\displaystyle\sum_{n=1}^{\infty}a_n = \lim_{n\to\infty}s_n = \lim_{n\to\infty}\frac{2n+1}{n+2} = 2.$

b. Notice that $a_5 = s_5 - s_4 = \frac{11}{7} - \frac{3}{2} = \frac{1}{14}$.

9. a. Since $|(\cos n)/n| < 1/n$ if $n \geqslant 1$ and $\lim(1/n) = 0$, it follows that $\lim(\cos n)/n = 0$. In this case, no conclusion can be drawn about the series only from the limit of its general term (although the series does converge by an extension of the alternating series test).

b. The long run behaviour of the sequence of terms of this series is not determined by its first eight terms. Here are three (among infinitely many) possibilities.

- If $a_n = 2^{-\frac{1}{2}(n+1)}\sin^2\frac{1}{2}\pi n + \cos^2\frac{1}{2}\pi n$ for $n \geqslant 1$, then $a_{2n} = 1$ for $n \geqslant 1$ and $\lim_{n\to\infty}a_n \neq 0$, so $\sum a_n$ diverges by the vanishing criterion.
- If $a_n = \dfrac{(|\pi e - n| + \pi e - n)(\sqrt{2^{n+1}}\cos^2\frac{1}{2}\pi n + \sin^2\frac{1}{2}\pi n)}{|\pi e - n|\sqrt{2^{n+3}}}$ for $n \geqslant 1$, then $\lim_{n\to\infty}a_n = 0$ (since $a_n = 0$ for $n \geqslant 9$) and no conclusion can be drawn about the series only from the limit of its general term (although the series does converge).
- If $a_n = \displaystyle\sum_{i=1}^{4}\left\{\prod_{\substack{1\leqslant j\leqslant 8\\ j\neq 2i-1}}\frac{2^{-i}(n-j)}{2i-j-1} + \prod_{\substack{1\leqslant j\leqslant 8\\ j\neq 2i}}\frac{n-j}{2i-j}\right\}$ for $n \geqslant 1$, then $\lim_{n\to\infty}a_n = \infty$ and $\sum a_n$ diverges by the vanishing criterion.

10. a. This is a geometric series with first term $-\frac{320}{27}$ and common ratio $-\frac{4}{9}$, so

$$\sum_{n=1}^{\infty}\frac{5(-4)^{n+2}}{3^{2n+1}} = \frac{-320/27}{1-(-4/9)} = -\tfrac{320}{39}.$$

b. This series diverges, since

$$\sum_{n=1}^{\infty}\log\frac{2n-1}{2n+1} = \lim_{n\to\infty}\sum_{k=1}^{n}\log\frac{2k-1}{2k+1} = -\lim_{n\to\infty}\log(2n+1) = -\infty.$$

11. a. Let $\alpha_n = (n!)^2/(2n!)$; then

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\frac{n+1}{2(2n+1)} = \tfrac{1}{4},$$

which is smaller than one. Therefore, $\sum\alpha_n$ converges by the ratio test.

b. Since

$$0 < \frac{\cos^2(k)}{k\sqrt{k}} < \frac{1}{k\sqrt{k}}$$

for $k \geqslant 1$,

$$\sum_{k=1}^{\infty}\frac{\cos^2(k)}{k\sqrt{k}}$$

converges with the p-series $\sum k^{-3/2}$ by the comparison test.

c. Since $e^x > x$ for all real numbers x, the series

$$\sum_{n=1}^{\infty}\frac{e^{\sqrt{n}}}{\sqrt{n}}$$

diverges by the vanishing condition (the general term diverges to ∞).

d. Since $\frac{1}{2}\vartheta < \vartheta\cos\vartheta < \sin\vartheta$ if $0 < \vartheta < \frac{1}{3}\pi$, $1/n < \sin(2/n)$ if $n \geqslant 2$, and so

$$\sum_{n=2}^{\infty}\sin\left(\frac{2}{n}\right)$$

diverges with the harmonic series by the comparison test.

12. a. Let $\alpha_n = \big(-n/(2n+1)\big)^{3n}$; then

$$\lim_{n\to\infty}\sqrt[n]{|\alpha_n|} = \lim_{n\to\infty}\left(\frac{1}{2+1/n}\right)^3 = \tfrac{1}{8},$$

which is less than one. Therefore, $\sum\alpha_n$ is absolutely convergent by the root test.

b. If $a_n = (\log n)/\sqrt{n}$ and $n \geqslant 3$ then $\log(n) > 1$ and thus $a_n > n^{-1/2} > 0$, so the comparison test implies that $\sum a_n$ diverges with the p-series $\sum n^{-1/2}$ ($p = \frac{1}{2} \leqslant 1$). On the other hand,

$$\frac{d}{dn}\left\{\frac{\log(n)}{\sqrt{n}}\right\} = \frac{2-\log(n)}{2n\sqrt{n}} < 0,$$

and hence $a_n > a_{n+1} > 0$ if $n > e^2$. Also, $\lim a_n = 0$ by elementary properties of the logarithm, so the Leibniz test implies that the series $\sum(-1)^n a_n$ is convergent. Therefore, $\sum(-1)^n a_n$ is conditionally convergent.

13. Let $\alpha_n = 3^n(x-2)^{n+1}/(2n+1)$; if $x \neq 3$ then

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \lim_{n\to\infty}\frac{3(2n+1)}{2n+3}|x-2| = 3|x-2|,$$

so by the ratio test $\sum\alpha_n$ converges absolutely if $\frac{5}{3} < x < \frac{7}{3}$ and diverges if $x < \frac{5}{3}$ or $x > \frac{7}{3}$. If $x = \frac{5}{3}$ then $\alpha_n = (-1)^{n+1}/(2n+1)$, so $\sum\alpha_n$ converges by the alternating series test (since $\{1/(2n+1)\}$ is positive and decreasing, and its limit is zero). If $x = \frac{7}{3}$ then $\alpha_n = 1/(2n+1) \geqslant 1/(3n)$ if $n \geqslant 1$, so $\sum\alpha_n$ diverges with the harmonic series by the comparison test. Therefore, the interval of convergence of $\sum\alpha_n$ is $[\,\frac{5}{3}, \frac{7}{3}\,)$; its radius of convergence is $\frac{1}{3}$.

14. The Maclaurin series of $\cos(t)$, with $t = 2x - \pi$, gives

$$\cos(2x) = -\cos(2x-\pi) = -\cos\big(2(x-\tfrac{1}{2}\pi)\big)$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}2^{2k}}{(2k)!}\big(x-\tfrac{1}{2}\pi\big)^{2k}$$

$$= -1 + 2\big(x-\tfrac{1}{2}\pi\big)^2 - \tfrac{2}{3}\big(x-\tfrac{1}{2}\pi\big)^4 + \tfrac{4}{45}\big(x-\tfrac{1}{2}\pi\big)^6 - \&c.$$