

- Given  $f(x) = 2x \arctan 2x - \frac{1}{2} \log(1 + 4x^2) + \arcsin \frac{2}{3}$ .
  - Find  $f'(x)$  and simplify your answer.
  - Evaluate  $f'(\frac{1}{2})$
- Evaluate each of the following limits, using  $\infty$  and  $-\infty$  when appropriate.
  - $\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{2x}$
  - $\lim_{x \rightarrow 0^+} (e^{-2/x} \log x)$
  - $\lim_{x \rightarrow 0} \frac{e^{6x} - 6x - 1}{x^2}$
- Evaluate each of the following integrals.
  - $\int \frac{2x + 1}{\sqrt{x - 3}} dx$
  - $\int \frac{9x - 1}{(x - 3)(x^2 + 4)} dx$
  - $\int x \operatorname{arcsec} x dx$
  - $\int_0^{\frac{1}{4}\pi} \sin^3 2x \cos^4 2x dx$
  - $\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1 - x^2}} dx$
  - $\int e^{3x} \sin x dx$
  - $\int \frac{dx}{\sqrt{9x^2 - 16}}$
- Evaluate each of the following improper integrals.
  - $\int_1^{\frac{2}{3}\sqrt{3}} \frac{dx}{x\sqrt{x^2 - 1}}$
  - $\int_4^{\infty} \frac{dx}{x \log x}$
- Solve the differential equation
 
$$2y \frac{dy}{dx} = y^2 - 1; \quad y(0) = 2.$$
- Sketch the region enclosed by  $y = 2/x - 1$  and  $y = 2 - x$ , and find its area.
- Let  $\mathcal{R}$  be the region enclosed by  $y = \sin x^2$  and the  $x$ -axis on  $[0, \sqrt{\pi}]$ .
  - Find the volume of the solid obtained by revolving  $\mathcal{R}$  about the  $y$ -axis.
  - Set up, but do not evaluate, an integral that represents the volume of the solid obtained by revolving  $\mathcal{R}$  about the  $x$ -axis.
- Determine whether the sequence converges or diverges; if it converges, find its limit.
  - $\{1 + \cos \frac{1}{2}(2n + 1)\pi\}$
  - $\left\{(-1)^n \frac{3n^2 + n - 2}{n^2}\right\}$

- Determine whether each statement is true or false. Justify each answer, with a proof or a counterexample, as appropriate.
  - If  $\lim |a_n| \neq 0$  then  $\lim a_n \neq 0$ .
  - If  $\lim a_n = 0$  then  $\sum_{n=1}^{\infty} \sin a_n$  converges.
- Find the sum of the series
 
$$\sum_{n=0}^{\infty} \frac{3^{n+1} + 2^n}{4^n}.$$
- Classify each of the following series as convergent or divergent, and justify your answers.
  - $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$
  - $\sum_{n=1}^{\infty} \left(\frac{2n - e}{n^2}\right)^{2n}$
  - $\sum_{n=1}^{\infty} \frac{\sqrt{n^3 - 1}}{n^2 + 1}$
  - $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$
- Classify each of the following series as absolutely convergent, conditionally convergent or divergent. Justify your answers.
  - $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^3 + 1}$
  - $\sum_{n=1}^{\infty} (-1)^n \cos \frac{1}{n}$
  - $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$
- Determine the radius and interval of convergence of the series
 
$$\sum_{n=1}^{\infty} \frac{3^{n-1}(x+1)^n}{n\sqrt{n+1}}.$$
- Let  $f(x) = \log(1 + x)$ .
  - Write the first five non-zero terms of the Maclaurin series of  $f$ .
  - Find a formula for the  $k^{\text{th}}$  term of the Maclaurin series, and write the series using sigma notation.

Solution outlines

- $f'(x) = 2 \arctan 2x + \frac{4x}{1 + 4x^2} - \frac{4x}{1 + 4x^2} + 0 = 2 \arctan 2x$ .
  - $f'(\frac{1}{2}) = 2 \arctan 1 = \frac{1}{2}\pi$ .
- One application of l'Hôpital's rule gives
 
$$\lim_{x \rightarrow \infty} \{2x \log(1 + 4/x)\} = 2 \lim_{t \rightarrow 0^+} \frac{\log(1 + 4t)}{t} = 8 \lim_{t \rightarrow 0^+} \frac{1}{1 + 4t} = 8,$$
 where  $t = 1/x$ , so the limit in question is equal to  $e^8$ .
  - One application of l'Hôpital's rule, after letting  $t = 1/x$ , gives
 
$$\lim_{x \rightarrow 0^+} (e^{-2/x} \log x) = - \lim_{t \rightarrow \infty} \frac{\log t}{e^{2t}} = - \lim_{t \rightarrow \infty} \frac{1}{2te^{2t}} = 0.$$
- Two applications of l'Hôpital's rule gives
 
$$\lim_{x \rightarrow 0} \frac{e^{6x} - 6x - 1}{x^2} = 3 \lim_{x \rightarrow 0} \frac{e^{6x} - 1}{x} = 18 \lim_{x \rightarrow 0} e^{6x} = 18.$$
- Repeated partial integration (integrating the the fractional power and differentiating the polynomial) gives
 
$$\int \frac{2x + 1}{\sqrt{x - 3}} dx = 2(2x + 1)\sqrt{x - 3} - \frac{8}{3}(x - 3)^{3/2} + C$$

$$= \frac{2}{3}(2x + 15)\sqrt{x - 3} + C.$$
  - Resolving the integrand into partial fractions and then integrating term by term yields
 
$$\int \frac{9x - 1}{(x - 3)(x^2 + 4)} dx = \int \left\{ \frac{2}{x - 3} - \frac{2x - 3}{x^2 + 4} \right\} dx$$

$$= \log \frac{(x - 3)^2}{x^2 + 4} + \frac{3}{2} \arctan \frac{1}{2}x + C.$$

- Partial integration gives
 
$$\int x \operatorname{arcsec} x dx = \frac{1}{2}x^2 \operatorname{arcsec} x - \frac{1}{2} \int \frac{x}{\sqrt{x^2 - 1}} dx$$

$$= \frac{1}{2}x^2 \operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2 - 1} + C.$$
- Changing the variable of integration to  $t = \cos(2x)$  gives
 
$$\int_0^{\frac{1}{4}\pi} \sin^3 2x \cos^4 2x dx = \frac{1}{2} \int_0^1 t^4(1 - t^2) dt = \frac{1}{70}t^5(7 - 5t^2) \Big|_0^1 = \frac{1}{35}.$$
- Changing the variable of integration to  $t = \arcsin x$  gives
 
$$\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1 - x^2}} dx = \int_0^{\frac{1}{6}\pi} t dt = \frac{1}{2}t^2 \Big|_0^{\frac{1}{6}\pi} = \frac{1}{72}\pi^2.$$
- Repeated partial integration (integrating the trigonometric function and differentiating the exponential function) gives
 
$$\int e^{3x} \sin x dx = -e^{3x} \cos x + 3e^{3x} \sin x - 9 \int e^{3x} \sin x dx,$$
 and therefore
 
$$\int e^{3x} \sin x dx = \frac{1}{10}e^{3x}(3 \sin x - \cos x) + C.$$
- Applying a standard integral formula gives
 
$$\int \frac{dx}{\sqrt{9x^2 - 16}} = \frac{1}{3} \log|3x + \sqrt{9x^2 - 16}| + C.$$

4. a. A standard integral formula gives

$$\int_1^{\frac{2}{3}\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec} \frac{2}{3}\sqrt{3} - \operatorname{arcsec} 1 = \frac{1}{6}\pi,$$

since  $\operatorname{arcsec}$  is continuous on  $[1, \frac{2}{3}\sqrt{3}]$ .

b. One has

$$\int_4^\infty \frac{dx}{x \log x} = \lim_{t \rightarrow \infty} \log \log t - \log \log 4 = \infty$$

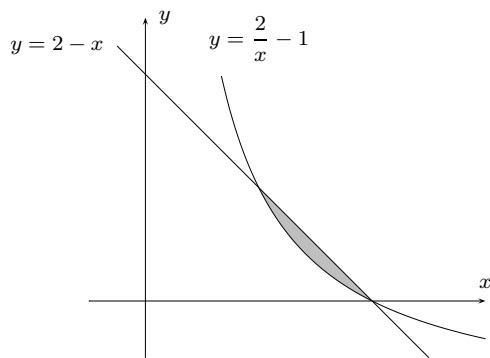
(so the integral diverges).

5. Separating variables and integrating gives

$$\int \frac{2y}{y^2-1} dy = \int dx, \text{ or } \log|y^2-1| = x + C, \text{ i.e., } y^2 = Ae^x + 1,$$

where  $A = \pm e^C$ . Now  $y(0) = 2$  gives  $A = 3$  and  $y > 1$ , and so  $y = \sqrt{3e^x + 1}$ .

6. Below is a sketch of the region in question.



The curves meet where  $2 - x = 2/x - 1$ , or  $0 = x^2 - 3x + 2 = (x - 1)(x - 2)$ , i.e., where  $x = 1$  or  $x = 2$ . On  $(1, 2)$  the line is above the hyperbola, so the area of the region in question is

$$\begin{aligned} \int_1^2 \{(2-x) - (2/x-1)\} dx &= \int_1^2 (3-x-2/x) dx \\ &= \left( 3x - \frac{1}{2}x^2 - 2 \log x \right) \Big|_1^2 \\ &= \frac{3}{2} - 2 \log 2. \end{aligned}$$

7. a. The solid obtained by revolving  $\mathcal{R}$  about the  $y$ -axis can be decomposed into cylindrical shells of radius  $x$  and height  $\sin x^2$ , for  $0 \leq x \leq \sqrt{\pi}$ , so its volume is equal to

$$2\pi \int_0^{\sqrt{\pi}} x \sin x^2 dx = -\pi \cos x^2 \Big|_0^{\sqrt{\pi}} = 2\pi.$$

b. The solid obtained by revolving  $\mathcal{R}$  about the  $x$ -axis can be decomposed into disks of radius  $\sin x^2$ , for  $0 \leq x \leq \sqrt{\pi}$ , so its volume is represented by the integral

$$\pi \int_0^{\sqrt{\pi}} \sin^2 x^2 dx.$$

8. a. Since  $\cos \frac{1}{2}(2n+1)\pi = 0$  for every natural number  $n$ , the given sequence converges to 1 (each of its terms is equal to 1).

b. Let  $a_n$  denote the general term of the given sequence. Since  $\lim a_{2n} = 3$  and  $\lim a_{2n+1} = -3$ , it follows that  $\{a_n\}$  has no limit.

9. a. This statement is true. For if  $\lim |a_n| \neq 0$ , there is a positive real number  $\varepsilon_0$  such that for any natural number  $N$  there is a natural number  $n \geq N$  for which  $||a_n| - 0| \geq \varepsilon_0$ , i.e.,  $|a_n - 0| \geq \varepsilon_0$ , which means that  $\lim a_n \neq 0$  by definition.

b. This statement is false. For example,

$$a_n = \arcsin \frac{1}{n} \rightarrow 0, \text{ and } \sum_{n=1}^{\infty} \sin a_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges.

10. The given series is the sum of two geometric series, and in fact

$$\sum_{n=0}^{\infty} \frac{3^{n+1} + 2^n}{4^n} = \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{3}{1-\frac{3}{4}} + \frac{1}{1-\frac{1}{2}} = 14.$$

11. a. The given series diverges because it is the difference of a divergent ( $p = 1$ ) and a convergent ( $p = 2$ )  $p$ -series. (Alternatively, the given series diverges with the harmonic series because its terms are larger than  $\frac{3}{4}n^{-1}$  if  $n > 2$ .)

b. Since

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{2n-e}{n^2} \right|^{2n}} = \lim_{n \rightarrow \infty} \frac{(2-e/n)^2}{n^2} = 0,$$

the series converges by the Root Test.

c. If  $n > 1$  then  $n^3 - 1 > \frac{1}{4}n^3$ ,  $n^2 + 1 < 2n^2$ , and therefore

$$\frac{\sqrt{n^3-1}}{n^2+1} > \frac{1}{4}n^{-1/2},$$

so the series in question diverges with  $\sum n^{-1/2}$  ( $p = \frac{1}{2}$ ) by the Comparison Test. (Alternatively, the Limit Comparison Test could be used.)

d. Since

$$\frac{(n!)^2}{(2n)!} = \frac{1}{2^n} \cdot \frac{n(n-1) \cdots 2 \cdot 1}{(2n-1)(2n-3) \cdots 3 \cdot 1} \leq \frac{1}{2^n},$$

the given series converges by the Comparison Test. (Alternatively, the Ratio test could be used.)

12. a. Since

$$0 < \frac{\arctan n}{n^3+1} < \frac{1}{2}\pi n^{-3}, \text{ for } n > 0,$$

the series in question is absolutely convergent by the Comparison Test.

b. Since  $\lim \cos \frac{1}{n} = 1$ , the series in question diverges by the vanishing criterion.

c. Let  $a_n = n/(n^2 + 1)$ . If  $n > 1$  then  $a_n > \frac{1}{2}n^{-1}$ , and so  $\sum (-1)^n a_n$  is not absolutely convergent by the Comparison Test. However,  $a_n > 0$ ,  $\{a_n\}$  is decreasing since

$$\frac{d}{dx} \left\{ \frac{x}{x^2+1} \right\} = \frac{1-x^2}{(x^2+1)^2} < 0 \text{ if } x > 1,$$

and

$$\lim a_n = \lim \frac{1}{n} \cdot \frac{1}{1+1/n^2} = 0,$$

so  $\sum (-1)^n a_n$  converges by the Alternating Series Test. Therefore,  $\sum (-1)^n a_n$  is conditionally convergent.

13. Let  $u_n$  denote the general term of the series in question. Then

$$\lim \left| \frac{u_{n+1}}{u_n} \right| = \frac{3|x+1|}{\sqrt{(1+1/n)(1+2/n)}} = 3|x+1|,$$

so  $\sum u_n$  is absolutely convergent if  $|x+1| < \frac{1}{3}$ , i.e.,  $-\frac{4}{3} < x < -\frac{2}{3}$ , by the Ratio Test. This means that the radius of convergence of  $\sum u_n$  is  $\frac{1}{3}$ . If  $x = -\frac{4}{3}$  or  $x = -\frac{2}{3}$  then

$$|u_n| = \frac{1}{n\sqrt{n+1}} < n^{-3/2},$$

and so  $\sum u_n$  is (absolutely) convergent by the Comparison Test. Therefore, the interval of convergence of  $\sum u_n$  is  $[-\frac{4}{3}, -\frac{2}{3}]$ .

14. We have

$$\begin{aligned} f(x) = \log(1+x) &= \int_0^x \frac{dt}{1+t} = \sum_{k=0}^{\infty} (-1)^k \int_0^x t^k dt \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \quad \text{(b.)} \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots \quad \text{(a.)} \end{aligned}$$