1. Find an equation of the line tangent to the graph of $y = x^2 \arcsin x$ at the point where $x = \frac{1}{2}$.

2. Evaluate the integrals.

a. $\displaystyle\int x^5\sqrt{x^3-1}\,dx$ b. $\displaystyle\int \tan^3(t)\sec^3(t)\,dt$

c. $\displaystyle\int_0^{\frac{1}{3}} \arctan 3x\,dx$ d. $\displaystyle\int \frac{x^3+x-2}{x^3+x}\,dx$

e. $\displaystyle\int \frac{x+5}{x^2+6x+13}\,dx$ f. $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

3. Evaluate the improper integrals.

a. $\displaystyle\int_2^\infty \frac{\operatorname{arcsec} x}{x\sqrt{x^2-1}}\,dx$ b. $\displaystyle\int_e^{e^2} \frac{\log x}{\sqrt{x\log x - x}}\,dx$

4. Evaluate the limits.

a. $\displaystyle\lim_{x\to 0}(1+\sin 2x)^{\cot x}$ b. $\displaystyle\lim_{x\to\frac{1}{2}\pi}\frac{\log(\sin x)}{(\pi-2x)^2}$ c. $\displaystyle\lim_{x\to 0^+}\csc(x)\arctan(x)$

5. Find the area of the region enclosed by $y = 2x^2$ and $y = x^2(x-1)$.

6. Let $\mathscr{R}$ be the region in Quadrant I between the x-axis and $y = \sqrt{4x^2 - x^4}$. Evaluate the volume of the solid obtained by revolving $\mathscr{R}$ about:
a. the y-axis; b. the line $y = -3$; c. the x-axis.

7. Give the explicit solution of the differential equation

$$e^{-x}y\frac{dy}{dx} = x; \quad y(0) = -1.$$

8. Does the sequence $\{n - \log n\}$ converge? If so, find its limit.

9. Determine whether each series converges or diverges.

a. $\displaystyle\sum_{n=1}^\infty\left\{\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right\}$ b. $\displaystyle\sum_{n=1}^\infty\left\{\frac{2}{\sqrt{n}} - \frac{1}{n^2}\right\}$ c. $\displaystyle\sum_{n=0}^\infty \frac{n^2 3^n}{(2n)!}$

10. Investigate (completely) the convergence of each series.

a. $\displaystyle\sum_{n=2}^\infty \frac{(-1)^n}{n\log\sqrt[3]{n}}$ b. $\displaystyle\sum_{n=1}^\infty (-1)^{n+1}\frac{\sin(1/n)}{\sqrt{n}}$

11. Determine whether each series *must*, *might*, or *cannot* converge, given only that $\lim a_k = 1$. Justify your answers.

a. $\displaystyle\sum_{k=1}^\infty (1-a_k)^k$ b. $\displaystyle\sum_{k=1}^\infty a_k$ c. $\displaystyle\sum_{k=1}^\infty \frac{a_k}{k}$ d. $\displaystyle\sum_{k=1}^\infty (-1)^k\frac{a_k}{k}$

12. Find the radius and interval of convergence of the power series

$$\sum_{n=0}^\infty \frac{2^{n+1}}{(n+1)!}(x-1)^n.$$

13. Find the Taylor series of $f(x) = x\log x$ centred at 1. For what values of x does this Taylor series converge?

14. You are given that there is a rational function $r(x)$ such that

$$r'(x) = \frac{p(x)}{x^2(x-3)^2},$$

where p is a polynomial of degree at most 3 such that $p(0) = p(3) = 3$.
a. Determine the resolution into partial fractions of $r'(x)$.
b. Calculate

$$\int \frac{p(x)}{x^2(x-3)^2}\,dx.$$

1. Since
$$y\Big|_{x=\frac{1}{2}} = x^2 \arcsin x\,\Big|_{x=\frac{1}{2}} = \tfrac{1}{24}\pi$$
and
$$\frac{dy}{dx}\Big|_{x=\frac{1}{2}} = \left(2x\arcsin x + \frac{x^2}{\sqrt{1-x^2}}\right)\Big|_{x=\frac{1}{2}} = \tfrac{1}{6}(\pi+\sqrt{3}),$$
it follows that $4(\pi+\sqrt{3})x - 24y = \pi + 2\sqrt{3}$ is an equation of the line tangent to the graph of $y = x^2 \arcsin x$ at the point where $x = \frac{1}{2}$.

2. a. Repeated partial integration (integrating x^2 times a fractional power of x^3-1 and differentiating the remaining power of x at each stage) yields
$$\begin{aligned}\int x^5\sqrt{x^3-1}\,dx &= \tfrac{2}{9}x^3(x^3-1)^{3/2} - \tfrac{2}{3}\int x^2(x^3-1)^{3/2}\,dx\\ &= \tfrac{2}{9}x^3(x^3-1)^{3/2} - \tfrac{4}{45}(x^3-1)^{5/2} + C\\ &= \tfrac{2}{45}(3x^3+2)\sqrt{(x^3-1)^3} + C.\end{aligned}$$
b. If $y = \sec t$ then $dy = \sec t\tan t\,dt$, $\tan^2 t = y^2 - 1$, and so
$$\begin{aligned}\int \tan^3(t)\,\sec^3(t)\,dt &= \int (y^2-1)y^2\,dy = \tfrac{1}{5}y^5 - \tfrac{1}{3}y^3 + C\\ &= \tfrac{1}{15}(3\sec^2 t - 5)\sec^3 t + C.\end{aligned}$$
c. Partial integration (integrating 1 and differentiating $\arctan 3x$) yields
$$\begin{aligned}\int_0^{\frac{1}{3}} \arctan 3x\,dx &= x\arctan 3x\Big|_0^{\frac{1}{3}} - \int_0^{\frac{1}{3}} \frac{3x}{9x^2+1}\,dx\\ &= \tfrac{1}{12}\pi - \tfrac{1}{6}\log(9x^2+1)\Big|_0^{\frac{1}{3}}\\ &= \tfrac{1}{12}\pi - \tfrac{1}{6}\log 2,\end{aligned}$$
where the rightmost integral in the first line is evaluated by mentally changing the variable of integration to $9x^2+1$.

d. After division, the proper rational function can be integrated by revising it and mentally changing the variable of integration to $1 + 1/x^2$, which gives
$$\begin{aligned}\int \frac{x^3+x-2}{x^3+x}\,dx &= \int\left\{1 - \frac{2}{x^3(1+1/x^2)}\right\}dx\\ &= x + \log(1+1/x^2) + C.\end{aligned}$$
e. Separating the numerator and completing the square in the denominator of the second term, gives
$$\begin{aligned}\int \frac{x+5}{x^2+6x+13}\,dx &= \int \frac{x+3}{x^2+6x+13}\,dx + 2\int\frac{dx}{(x+3)^2+4}\\ &= \tfrac{1}{2}\log(x^2+6x+13) + \arctan\tfrac{1}{2}(x+3) + C.\end{aligned}$$
f. Partial integration (integrating the negative power of x and differentiating the radical expression) yields
$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= -\frac{\sqrt{4-x^2}}{x} - \int\frac{dx}{\sqrt{4-x^2}}\\ &= -\frac{\sqrt{4-x^2}}{x} - \arcsin\tfrac{1}{2}x + C.\end{aligned}$$

3. a. By inspection, $\frac{1}{2}(\operatorname{arcsec} x)^2$ is an antiderivative of the integrand, so
$$\begin{aligned}\int_2^\infty \frac{\operatorname{arcsec} x}{x\sqrt{x^2-1}}\,dx &= \tfrac{1}{2}\big\{\lim_{t\to\infty}(\operatorname{arcsec} t)^2 - (\operatorname{arcsec} 2)^2\big\}\\ &= \tfrac{1}{2}\big\{\big(\tfrac{1}{2}\pi\big)^2 - \big(\tfrac{1}{3}\pi\big)^2\big\}\\ &= \tfrac{5}{72}\pi^2.\end{aligned}$$
b. Since $2\sqrt{x\log x - x}$ is a continuous antiderivative of the integrand on $[\,e, e^2\,]$,
$$\begin{aligned}\int_e^{e^2} \frac{\log x}{\sqrt{x\log x - x}}\,dx &= 2\sqrt{x\log x - x}\,\Big|_e^{e^2}\\ &= 2e.\end{aligned}$$

4. a. Since $\lim\limits_{x\to 0}(1+\sin 2x)^{1/(\sin 2x)} = e$, and
$$\lim_{x\to 0}\{\sin(2x)\cot(x)\} = \lim_{x\to 0}\frac{2\sin(x)\cos^2(x)}{\sin(x)} = 2,$$
it follows that $\lim\limits_{x\to 0}(1+\sin 2x)^{\cot x} = e^2$.

b. Elementary properties of the logarithm imply that
$$\lim_{x\to\frac{1}{2}\pi}\frac{\log(\sin x)}{\sin(x)-1} = 1.$$
If $2y = \pi - 2x$ then $\sin(x) - 1 = \cos(y) - 1 = -2\sin^2\big(\tfrac{1}{2}y\big)$, and
$$\lim_{x\to\frac{1}{2}\pi}\frac{\sin(x)-1}{(\pi-2x)^2} = -\tfrac{1}{8}\lim_{y\to 0}\left(\frac{\sin\big(\frac{1}{2}y\big)}{\frac{1}{2}y}\right)^2 = -\tfrac{1}{8}.$$
Therefore, the limit in question is equal to $-\frac{1}{8}$.

c. One application of l'Hôpital's Rule gives
$$\lim_{x\to 0^+}\csc(x)\arctan(x) = \lim_{x\to 0^+}\frac{\arctan x}{\sin x} = \lim_{x\to 0^+}\frac{1}{(1+x^2)\cos x} = 1.$$

5. The curves meet where $x^2(x-1) = 2x^2$, or $x^2(x-3) = 0$, *i.e.*, where $x = 0$ or $x = 3$. On $(\,0, 3\,)$ the cubic is smaller than the quadratic, so the area of the region enclosed by the given curves is equal to
$$\int_0^3\big\{(2x^2 - x^2(x-1)\big\}\,dx = \int_0^3(3x^2 - x^3)\,dx = \big(x^3 - \tfrac{1}{4}x^4\big)\Big|_0^3 = \tfrac{27}{4}.$$

6. If $x \geqslant 0$ then $y = \sqrt{4x^2 - x^4} = x\sqrt{4-x^2}$ meets the x-axis where $x = 0$ or $x = 2$, and the region $\mathscr{R}$ is above the x-axis and below the curve for $0 < x < 2$.

a. The solid obtained by revolving $\mathscr{R}$ about the y-axis can be decomposed into cylindrical shells of radius x and height $x\sqrt{4-x^2}$, for $0 \leqslant x \leqslant 2$; therefore, its volume is equal to
$$2\pi\int_0^2 x^2\sqrt{4-x^2}\,dx.$$
To evaluate the integral, first subtract and add 4 to expand the integrand in powers of $4 - x^2$; then, partial integration yields
$$\begin{aligned}\int_0^2 x^2\sqrt{4-x^2}\,dx &= -\int_0^2(4-x^2)^{3/2}\,dx + 4\int_0^2\sqrt{4-x^2}\,dx\\ &= -x(4-x^2)^{3/2}\,\Big|_0^2 - 3\int_0^2 x^2\sqrt{4-x^2}\,dx + 4\int_0^2\sqrt{4-x^2}\,dx\\ &= \int_0^2\sqrt{4-x^2}\,dx,\end{aligned}$$
which is π (since the last integral is one-quarter the area of a circle of radius 2). Therefore, the volume in question is equal to $2\pi^2$.

b. The solid obtained by revolving $\mathscr{R}$ about the line $y = 3$ can be decomposed into annuli of inner radius 3 and outer radius $3 + x\sqrt{4-x^2}$, for $0 \leqslant x \leqslant 2$; therefore, its volume is equal to
$$\begin{aligned}\pi\int_0^2\big\{\big(3 + x\sqrt{4-x^2}\,\big)^2 - 9\big\}\,dx &= \pi\int_0^2\big\{6x\sqrt{4-x^2} + 4x^2 - x^4\big\}\,dx\\ &= \pi\big\{-2(4-x^2)^{3/2} + \tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\big\}\Big|_0^2 = \tfrac{304}{15}\pi.\end{aligned}$$
c. The solid obtained by revolving $\mathscr{R}$ about the x-axis can be decomposed into disks of radius $x\sqrt{4-x^2}$, for $0 \leqslant x \leqslant 2$; therefore its volume is equal to
$$\pi\int_0^2(4x^2 - x^4)\,dx = \pi\big(\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\big)\Big|_0^2 = \tfrac{64}{15}\pi.$$

7. Separating the variables and integrating (after doubling) yields
$$\int 2y\,dy = 2\int xe^x\,dx, \quad\text{or}\quad y^2 = 2e^x(x-1) + C.$$
Now $y(0) = -1$ gives $1 = -2 + C$, or $C = 3$, and $y < 0$, so the explicit solution of the given differential equation is $y = -\sqrt{2e^x(x-1)+3}$.

8. Elementary properties of the logarithm imply that
$$\lim\{n - \log n\} = \lim\log(e^n/n) = \lim n = \infty,$$
so the sequence diverges to ∞.

9. a. Since

$$\Big(1-\frac{1}{\sqrt{2}}\Big)+\Big(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\Big)+\cdots+\Big(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\Big) = 1-\frac{1}{\sqrt{n+1}},$$

it follows that the series converges to 1.

b. Since

$$\frac{2}{\sqrt{n}}-\frac{1}{n^2}\geqslant\frac{1}{\sqrt{n}}>0$$

if $n\geqslant 1$, the given series diverges with the p-series $\sum n^{-1/2}$ by the Comparison Test. (Alternatively, this is the difference of a divergent series and a convergent series, which therefore diverges.)

c. Since

$$\lim\frac{(n+1)^2 3^{n+1}}{(2(n+1))!}\cdot\frac{(2n)!}{n^2 3^n}=\lim\frac{1}{n^2}\cdot\frac{3(1+1/n)}{2(2+1/n)}=0<1,$$

the series in question converges by the ratio test.

10. a. Since

$$\frac{1}{n\log\sqrt[3]{n}}=\frac{1}{3}\cdot\frac{1}{n\log n}$$

if $n\geqslant 2$, the given series is a non-zero multiple of a conditionally convergent alternating logarithmic p-series ($p=1$), and is therefore conditionally convergent.

b. Since

$$0<\frac{\sin(1/n)}{\sqrt{n}}<\frac{1}{n\sqrt{n}}$$

if $n\geqslant 1$, and $\sum n^{-3/2}$ is a convergent p-series, the given series is absolutely convergent by the comparison test.

11. a. Since $\lim\sqrt[k]{|1-a_k|^k}=\lim|1-a_k|=1-1=0$, which is less than 1, the series $\sum(1-a_k)^k$ *must* converge (absolutely) by the root test.

b. Since $\lim a_k=1\neq 0$, $\sum a_k$ *cannot* converge by the vanishing criterion.

c. Since $\lim(a_k/k)/(1/k)=\lim a_k=1$, and the harmonic series $\sum 1/k$ is divergent, the comparison test implies that $\sum a_k/k$ *cannot* converge.

d. If $a_k=1$ for $k\geqslant 1$, then $\sum(-1)^k a_k/k$ converges by the alternating series test (because $\{1/k\}$ is positive, decreasing and converges to zero). On the other hand, if $a_1=0$ and $a_k=1+(-1)^k/\log k$ for $k\geqslant 2$, then $\lim a_k=1$, but

$$(-1)^k\frac{a_k}{k}=\frac{(-1)^k}{k}+\frac{1}{k\log k}$$

for $k\geqslant 2$, and so $\sum(-1)^k a_k/k$ diverges because it is the sum of a convergent alternating series (as was just shown) and a divergent ($p=1$) logarithmic p-series. Therefore, the series $\sum(-1)^k a_k$ *might* converge.

12. If $x\neq 1$ and

$$u_n=\frac{2^{n+1}}{(n+1)!}(x-1)^n \quad\text{then}\quad \lim\left|\frac{u_{n+1}}{u_n}\right|=|x-1|\lim\frac{2}{n+2}=0,$$

so by the ratio test $\sum u_n$ converges for all real values of x (its radius is infinite).

13. The Maclaurin series of $\log(1+t)$ for $-1<t\leqslant 1$, where $t=x-1$, gives

$$\begin{aligned}
x\log x &= \big\{1+(x-1)\big\}\log\big\{1+(x-1)\big\}\\
&= \big\{1+(x-1)\big\}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n\\
&= (x-1)+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n+\sum_{n=2}^{\infty}\frac{(-1)^n}{n-1}(x-1)^n\\
&= (x-1)+\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}(x-1)^n,
\end{aligned}$$

which is valid *at least* for $0<x\leqslant 2$. Moreover, if α_n denotes the general term of this series when $x=0$, then

$$0<\alpha_{n+1}=\frac{1}{n(n+1)}<\frac{1}{n^2}$$

if $n\geqslant 2$, so $\sum\alpha_n$ converges with the p-series $\sum n^{-2}$ by the comparison test. Therefore, the interval of convergence of Taylor series in question is $[\,0,2\,]$.

14. a. The resolution into partial fractions of $r'(x)$ has the form

$$\frac{p(x)}{x^2(x-3)^3}=\frac{A}{x^2}+\frac{B}{(x-3)^2},$$

since it is the derivative of a rational function, and so

$$p(x)=A(x-3)^2+Bx^2.$$

Hence, $3=p(0)=9A$, so $A=\frac{1}{3}$, and $3=p(3)=9B$, so $B=\frac{1}{3}$. Therefore,

$$r'(x)=\frac{p(x)}{x^2(x-3)^3}=\frac{1}{3}\left\{\frac{1}{x^2}+\frac{1}{(x-3)^2}\right\}.$$

b. From part a, it follows that

$$\begin{aligned}
\int\frac{p(x)}{x^2(x-3)^3}\,dx &= \frac{1}{3}\int\left\{\frac{1}{x^2}+\frac{1}{(x-3)^2}\right\}dx\\
&= -\frac{1}{3}\left\{\frac{1}{x}+\frac{1}{x-3}\right\}+C.
\end{aligned}$$