1. Find the derivative with respect to x of

$$y = \arcsin\sqrt{1-x^2} + \operatorname{arccot}(1/x) - \operatorname{arcsec}(5).$$

Assuming that x is positive, simplify your answer as much as possible.

2. Evaluate each of the following limits.

a. $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\log x}\right)$ b. $\displaystyle\lim_{x\to 0}\frac{3x}{\arctan 3x}$ c. $\displaystyle\lim_{x\to \frac{1}{2}\pi^-}(\tan x)^{\cos x}$

3. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{x+1}{\sqrt{2x+1}}\,dx$ b. $\displaystyle\int_0^{\frac{1}{4}\pi}(1+\tan x)^3\sec^2 x\,dx$

c. $\displaystyle\int \sqrt{x^5}\log x\,dx$ d. $\displaystyle\int \frac{\sec^4 x}{\tan^2 x}\,dx$

e. $\displaystyle\int \frac{x^2+x}{x^4-1}\,dx$ f. $\displaystyle\int \frac{x^2}{\sqrt{1-4x^2}}\,dx$

4. Determine whether or not the integral converges. If it converges, compute its exact value.

a. $\displaystyle\int_0^9 \frac{dx}{\sqrt[3]{1-x}}$ b. $\displaystyle\int_{-\infty}^0 \frac{dx}{1+4x^2}$

5. Solve the differential equation. Give implicit *and* explicit solutions.

$$\frac{dy}{dx}\sqrt{1-x^2} = xy; \qquad y(0)=1,\ \ y>0.$$

6. Compute the exact area of the region bounded by the graphs of $y = x$ and $y = \log x$ on the interval $[\,1, e^2\,]$.

7. Let $\mathscr{R}$ be the region enclosed by the graph of $x + y^2 + 2y = 0$ and the y-axis. Find the volume of the solid obtained by revolving $\mathscr{R}$ about a. the x-axis, b. the y-axis, c. the line with equation $y = 5$.

8. Determine whether each sequence converges or diverges. If a sequence converges, find its limit.

a. $\left\{\dfrac{3^n}{n!}\right\}$ b. $\{n2^{1/n}\}$ c. $\left\{\dfrac{1+\cos n}{\sqrt{n}}\right\}$

9. Let

$$\sum_{n=1}^{\infty} a_n$$

be the series whose n^{th} partial sum is

$$s_n = \frac{2n+1}{3n^2-n}.$$

a. Determine whether the series converges or diverges. If it converges find its sum.

b. Find a_3.

10. Provide an example of each of the following.

a. A sequence $\{a_n\}$ such that $\lim a_n = 0$, yet $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

b. A convergent series $\displaystyle\sum_{n=1}^{\infty}(a_n+b_n)$, such that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverge.

c. A function f such that $\displaystyle\lim_{x\to\infty} f(x)$ is undefined, but $\lim f(n)$ is defined.

d. A continuous function $f(x)$ on $[\,0,\infty\,)$ such that $\displaystyle\int_0^{\infty} f(x)\,dx$ converges, but $\displaystyle\lim_{x\to\infty} f(x)$ is not equal to zero.

11. Suppose that f and g are decreasing functions such that $0 < f(x) < g(x)$ for $x \geqslant 1$, and that

$$\sum_{n=9}^{\infty} f(n)$$

converges. Fill in each blank with the appropriate word. The appropriate word is *must*, *might*, or *cannot*.

a. The series $\displaystyle\sum_{n=1}^{\infty} f(n)$ ________ converge.

b. The integral $\displaystyle\int_1^{\infty} f(x)\,dx$ ________ converge.

c. The series $\displaystyle\sum_{n=1}^{\infty} g(n)$ ________ converge.

d. The series $\displaystyle\sum_{n=9}^{\infty} \frac{1}{f(n)}$ ________ converge.

e. If $\displaystyle\lim_{n\to\infty}\frac{g(n)}{f(n)} = 5^8$ then $\displaystyle\sum_{n=9}^{\infty} g(n)$ ________ converge.

12. Determine whether the series converges or diverges. Justify your answers.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan n}{\sqrt{n+1}}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{(n+1)^2}\cos\left(\frac{\pi}{4n}\right)$ c. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3k}{k^3+1} - \frac{2^{k-1}}{3^k}\right)$

13. Determine whether the series is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n n^2}{(2n)!}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos\big((n+1)\pi\big)}{n^{1/3}}$

c. $\displaystyle\sum_{n=1}^{\infty}(-4)^n\left(2-\frac{1}{n}\right)^{2n}$ d. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(2^n)}{(n+1)(n+2)}$

14. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{(x-2)^n}{2^n n}.$$

15. Find the Maclaurin series of $f(x) = e^{-2x}$.

16. Find the Taylor series of $f(x) = (2x-1)^{-2}$ centred at 1. Write the first five terms of this series explicitly.

1. The formulæ for the derivatives of the inverse sine and inverse cotangent functions, combined with the Chain Rule, gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-(1-x^2)}}\cdot\frac{-x}{\sqrt{1-x^2}} + \frac{-1}{1+1/x^2}\cdot\frac{-1}{x^2} \\ &= \frac{-1}{\sqrt{1-x^2}} + \frac{1}{1+x^2},\end{aligned}$$

since $\sqrt{x^2} = x$ on $(0,1)$.

2. a. Combining terms, dividing, and then expanding about 1, gives

$$\begin{aligned}\lim_{x\to1}\left(\frac{x}{x-1} - \frac{1}{\log x}\right) &= \lim_{x\to1}\frac{(x-1)\log x + \log x - (x-1)}{(x-1)\log x} \\ &= 1 + \lim_{x\to1}\frac{-\frac12 + \frac13(x-1) - \frac14(x-1)^2 + \cdots}{1 - \frac12(x-1) + \frac13(x-1)^2 - \cdots} \\ &= \tfrac12.\end{aligned}$$

b. One application of l'Hôpital's Rule gives

$$\lim_{x\to0}\frac{3x}{\arctan 3x} = \lim_{x\to0}\frac{3(1+9x^2)}{3} = 1.$$

c. Writing the expression in the limit in term of the exponential function gives

$$\begin{aligned}\lim_{x\to\frac12\pi^-}(\tan x)^{\cos x} &= \lim_{x\to\frac12\pi^-} e^{\cos x\log(\tan x)} \\ &= e^0 \\ &= 1,\end{aligned}$$

since

$$\begin{aligned}\lim_{x\to\frac12\pi^-}\{\cos x\log(\tan x)\} &= \lim_{x\to\frac12\pi^-}\frac{\log(\tan x)}{\sec x} \\ &= \lim_{x\to\frac12\pi^-}\frac{\sec^2 x}{\sec x\tan^2 x} \\ &= \lim_{x\to\frac12\pi^-}\frac{\cos x}{\sin^2 x} \\ &= 0,\end{aligned}$$

by l'Hôpital's Rule.

3. a. Partial integration (integrating the power of $2x+1$ and differentiating the polynomial) gives

$$\begin{aligned}\int\frac{x+1}{\sqrt{2x+1}}\,dx &= (x+1)\sqrt{2x+1} - \int\sqrt{2x+1}\,dx \\ &= (x+1)\sqrt{2x+1} - \tfrac13\sqrt{(2x+1)^3} + C \\ &= \tfrac13(x+2)\sqrt{2x+1} + C.\end{aligned}$$

b. If $t = 1+\tan x$ then $dt = \sec^2 x\,dx$, $t=1$ if $x=0$, $t=2$ if $x=\frac14$, and so

$$\int_0^{\frac14\pi}(1+\tan x)^3\sec^2 x\,dx = \int_1^2 t^3\,dt = \tfrac14 t^4\Big|_1^2 = \tfrac{15}{4}.$$

c. Partial integration (integrating the power function and differentiating the logarithm) gives

$$\begin{aligned}\int\sqrt{x^5}\log x\,dx &= \tfrac27 x^{7/2}\log x - \tfrac27\int x^{5/2}\,dx \\ &= \tfrac27 x^{7/2}\log x - \tfrac{4}{49}x^{7/2} + C \\ &= \tfrac{2}{49}(7\log x - 2)\sqrt{x^7} + C.\end{aligned}$$

d. If $t = \tan x$ then $dt = \sec^2 x\,dx$, $\sec^2 x = \tan^2 x + 1$, and so

$$\int\frac{\sec^4 x}{\tan^2 x}\,dx = \int(t^2+1)/t^2\,dt = t - t^{-1} + C = \tan x - \cot x + C.$$

e. Simplifying the integrand, and resolving into partial fractions, gives

$$\frac{x^2+x}{x^4-1} = \frac{x}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} + \frac{-x+1}{2(x^2+1)}.$$

Therefore,

$$\begin{aligned}\int\frac{x^2+x}{x^4-1}\,dx &= \tfrac12\int\left\{\frac{1}{x-1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right\}dx \\ &= \tfrac12\log|x-1| - \tfrac14\log(x^2+1) + \tfrac12\arctan x + C \\ &= \tfrac14\log\frac{(x-1)^2}{x^2+1} + \tfrac12\arctan x + C\end{aligned}$$

f. Integrating by parts and then rationalizing the numerator of the remaining integrand, yields

$$\begin{aligned}\int\frac{x^2}{\sqrt{1-4x^2}}\,dx &= -\tfrac14 x\sqrt{1-4x^2} + \tfrac14\int\sqrt{1-4x^2}\,dx \\ &= -\tfrac14 x\sqrt{1-4x^2} + \tfrac14\int\frac{dx}{\sqrt{1-4x^2}} - \frac{x^2}{\sqrt{1-4x^2}}\,dx \\ &= -\tfrac18 x\sqrt{1-4x^2} + \tfrac{1}{16}\arcsin 2x + C.\end{aligned}$$

4. a. The integrand has an infinite discontinuity at 1, and so

$$\begin{aligned}\int_0^9\frac{dx}{\sqrt[3]{(1-x)}}\,dx &= \int_0^1\frac{dx}{\sqrt[3]{(1-x)}}\,dx + \int_1^9\frac{dx}{\sqrt[3]{(1-x)}}\,dx \\ &= -\tfrac32\lim_{\alpha\to1^-}\sqrt[3]{(1-x)^2}\Big|_0^\alpha - \tfrac32\lim_{\beta\to1^+}\sqrt[3]{(1-x)^2}\Big|_\beta^9 \\ &= -\tfrac92.\end{aligned}$$

Note: If an antiderivative of the integrand is continuous on the bounded interval of integration of an improper integral, then the improper integral can be calculated as an ordinary definite integral. If you **cite and state** this theorem, then it legitimate to evaluate the improper integral as follows.

$$\int_0^9\frac{dx}{\sqrt[3]{1-x}}\,dx = -\tfrac32\sqrt[3]{(1-x)^2}\Big|_0^9 = -\tfrac92,$$

because the antiderivative $-\frac32\sqrt[3]{(1-x)^2}$ *of the integrand is continuous on* $[0,9]$.

b. This is improper integral of a continuous function on $(-\infty,0]$, and so

$$\int_{-\infty}^0\frac{dx}{1+4x^2}\,dx = \tfrac12\lim_{s\to-\infty}\arctan(2x)\Big|_s^0 = \tfrac14\pi.$$

5. Separating the variables gives

$$\frac1y\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}, \qquad \text{and so} \qquad \log|y| = C - \sqrt{1-x^2}.$$

The condition $y(0)=1$ gives $0 = \log 1 = C - \sqrt{1-0^2} = C-1$, or $C=1$. Therefore,

$$\log|y| = 1 - \sqrt{1-x^2}, \qquad \text{or} \qquad y = e^{1-\sqrt{1-x^2}}.$$

6. Let $f(x) = x$ and $g(x) = \log x$; then $f(1) = 1 > 0 = g(1)$, and $f'(x) = 1 > 1/x = g'(x)$ for $x > 1$, so it follows (using a corollary of the Mean Value Theorem from Calculus I) that $f(x) > g(x)$ for $x > 1$ (and therefore in fact for $x > 0$, since $\log x < 0$ if $0 < x < 1$). Therefore, the area in question is equal to

$$\int_1^{e^2}(x - \log x)\,dx = \left\{\tfrac12x^2 - x(\log x - 1)\right\}\Big|_1^{e^2} = \tfrac12e^4 - e^2 - \tfrac32.$$

7. The graph of $x + y^2 + 2y = 0$ meets the y-axis where $x = 0$, *i.e.*, where $0 = y^2 + 2y = y(y+2)$, or $y = 0, -2$. Therefore,

$$\mathscr{R} = \{\,(x,y)\colon\ -2 \leqslant y \leqslant 0 \text{ and } 0 \leqslant x \leqslant -(y^2+2y)\,\}.$$

a. The solid obtained by revolving $\mathscr{R}$ about the x-axis can be decomposed into cylindrical shells of radius $-y$ and height $-(y^2+2y)$, for $-2 \leqslant y \leqslant 0$. Therefore, the volume of the solid in question is equal to

$$2\pi\int_{-2}^0 y(y^2+2y)\,dy = 2\pi\left(\tfrac14y^4 + \tfrac23y^3\right)\Big|_{-2}^0 = \tfrac83\pi.$$

b. The the solid obtained by revolving $\mathscr{R}$ about the y-axis can be decomposed into circular disks of radius $-(y^2+2y)$, for $-2 \leqslant y \leqslant 0$. So the volume of the solid in question is equal to

$$\begin{aligned}\pi\int_{-2}^{0}(y^2+2y)^2\,dy &= \pi\int_{-2}^{0}(y^4+4y^3+4y^2)\,dy\\ &= \pi\big(\tfrac{1}{5}y^5+y^4+\tfrac{4}{3}y^3\big)\Big|_{-2}^{0}\\ &= \tfrac{16}{15}\pi.\end{aligned}$$

c. The solid obtained by revolving $\mathscr{R}$ about the line with equation $y=5$ can be decomposed into cylindrical shells of radius $5-y$ and height $-(y^2+2y)$, for $-2 \leqslant y \leqslant 0$. Therefore, the volume of the solid in question is equal to

$$\begin{aligned}2\pi\int_{-2}^{0}(y-5)(y^2+2y)\,dy &= 2\pi\int_{-2}^{0}(y^3-3y^2-10y)\,dy\\ &= 2\pi\big(\tfrac{1}{4}y^4-y^3-5y^2\big)\Big|_{-2}^{0}\\ &= 16\pi.\end{aligned}$$

8. a. Since

$$\sum_{n=0}^{\infty}\frac{3^n}{n!}=e^3,$$

the *sequence* $\{3^n/n!\}$ converges to zero by the Vanishing Criterion.

b. Since $\lim 2^{1/n}=1$, and $\lim n=\infty$, the sequence $\{n2^{1/n}\}$ diverges to ∞.

c. Since

$$0<\frac{1+\cos n}{\sqrt{n}}<\frac{2}{\sqrt{n}},$$

for $n \geqslant 1$ and $\lim\{2/\sqrt{n}\}=0$, the sequence $\big\{(1+\cos n)/\sqrt{n}\big\}$ converges to zero by the Squeeze Theorem.

9. a. Since

$$\lim s_n=\lim\frac{2n+1}{3n^2-n}=\lim\left\{\frac{1}{n}\cdot\frac{2+1/n}{3+1/n}\right\}=0,$$

the series in question converges to zero (*i.e.*, its sum is zero).

b. Since $s_3=a_1+a_2+a_3$ and $s_2=a_1+a_2$, it follows that

$$a_3=s_3-s_2=\frac{2\cdot 3+1}{3\cdot 3^2-3}-\frac{2\cdot 2+1}{3\cdot 2^2-2}=-\tfrac{5}{24}.$$

10. a. If $a_n=\log(1+1/n)$ then $\lim a_n=0$, yet $\sum a_n$ diverges, because its sequence of partial sums,

$$\sum_{i=1}^{n}a_i=\sum_{i=1}^{n}\big\{\log(i+1)-\log i\big\}=\log(n+1),$$

diverges to ∞.

b. If $a_n=1$ and $b_n=-1$ for all $n \geqslant 1$ then $\sum a_n$ and $\sum b_n$ diverge by the Vanishing Criterion, but $a_n+b_n=0$ for all $n \geqslant 1$, so $\sum(a_n+b_n)$ converges.

c. If $f(x)=\sin\pi x$, then $\lim\limits_{x\to\infty}f(x)$ is undefined (because f assumes all values in $[-1,1]$ on any ray of the form (a,∞), where a is a real number), but $\lim f(n)=\lim\sin\pi n=0$, since $\sin\pi n=0$ for any integer n.

d. If $f(x)=\sin\big(\frac{1}{2}x^2\big)$ then $\lim\limits_{x\to\infty}f(x)$ is undefined (since f assumes all values in $[-1,1]$ on any ray of the form (a,∞) where a is a real number). On the other hand, partial integration and the Squeeze theorem yield

$$\begin{aligned}\int_0^{\infty}f(x)\,dx &= \int_0^{\sqrt{\pi}}\sin\big(\tfrac{1}{2}x^2\big)\,dx+\int_{\sqrt{\pi}}^{\infty}\sin\big(\tfrac{1}{2}x^2\big)\,dx\\ &= \int_0^{\sqrt{\pi}}\sin\big(\tfrac{1}{2}x^2\big)\,dx-\lim_{t\to\infty}\frac{\cos\big(\frac{1}{2}x^2\big)}{x}\bigg|_{\sqrt{\pi}}^{t}-\int_{\sqrt{\pi}}^{\infty}\frac{\cos\big(\frac{1}{2}x^2\big)}{x^2}\,dx\\ &= \int_0^{\sqrt{\pi}}\sin\big(\tfrac{1}{2}x^2\big)\,dx-\int_{\sqrt{\pi}}^{\infty}\frac{\cos\big(\frac{1}{2}x^2\big)}{x^2}\,dx,\end{aligned}$$

and since

$$\left|\frac{\cos\big(\frac{1}{2}x^2\big)}{x^2}\right|<\frac{1}{x^2}$$

on $[\sqrt{\pi},\infty)$, and

$$\int_{\sqrt{\pi}}^{\infty}\frac{dx}{x^2}=\lim_{t\to\infty}\left(-\frac{1}{x}\right)\bigg|_{\sqrt{\pi}}^{t}=\frac{1}{\sqrt{\pi}}$$

the Comparison Principle implies that

$$\int_{\sqrt{\pi}}^{\infty}\frac{\cos\big(\frac{1}{2}x^2\big)}{x^2}\,dx$$

converges. Therefore, $\displaystyle\int_0^{\infty}f(x)\,dx$ converges.

11. a. Since

$$\sum_{n=1}^{\infty}f(n)=\sum_{n=1}^{8}f(n)+\sum_{n=9}^{\infty}f(n),$$

the series

$$\sum_{n=1}^{\infty}f(n)$$

must converge.

b. The difficult part of this problem is to show that the Riemann integral of f on $[1,\alpha]$ is defined whenever $1<\alpha$, since we are only given that f is decreasing and not that f is continuous. Let $P=\{x_0,x_1,\dots,x_n\}$ be any partition of $[1,\alpha]$ such that $0<x_i-x_{i-1}<(\alpha-1)/n$ for $i=1,\dots,n$. If $\{\xi_1,\dots,\xi_n\}$ is any list of numbers such that $x_{i-1}\leqslant\xi_i\leqslant x_i$ for $i=1,\dots,n$ then the corresponding Riemann sum satisfies

$$\sum_{i=1}^{n}f(x_i)(x_i-x_{i-1})\leqslant\sum_{i=1}^{n}f(\xi_i)(x_i-x_{i-1})\leqslant\sum_{i=1}^{n}f(x_{i-1})(x_i-x_{i-1})$$

since f is decreasing, and

$$\begin{aligned}0<\sum_{i=1}^{n}f(x_{i-1})(x_i-x_{i-1})-\sum_{i=1}^{n}f(x_i)(x_i-x_{i-1})&\\ <\frac{\alpha-1}{n}\sum_{i=1}^{n}\big(f(x_{i-1})-f(x_i)\big)&\\ =\frac{\alpha-1}{n}\big(f(1)-f(\alpha)\big),&\end{aligned}$$

which can be made smaller than any positive number by choosing n sufficiently large. Therefore, the Riemann integral of f on $[1,\alpha]$ is defined and is equal to the least upper bound of all lower sums

$$\sum_{i=1}^{n}f(x_i)(x_i-x_{i-1}),$$

and to the greatest lower bound of all upper sums

$$\sum_{i=1}^{n}f(x_{i-1})(x_i-x_{i-1}),$$

for partitions $P=\{x_0,x_1,\dots,x_n\}$ of $[1,\alpha]$. (In fact, the same argument shows that the Riemann integral of f on $[a,b]$ for any real numbers a, and b such that $1\leqslant a<b$.) The rest of the solution is similar to part of the proof of the Integral Test. Since f is positive and decreasing on $[1,\infty)$,

$$0<\int_1^{\alpha}f(x)\,dx<\int_1^{\beta}f(x)\,dx<\sum_{n=1}^{k}f(n)$$

whenever $1<\alpha<\beta\leqslant k$. Since it is given that the series $\sum f(n)$ converges, it follows that

$$\left\{\int_1^{\alpha}f(x)\,dx\colon 1<\alpha\right\}$$

is bounded, and that its least upper bound is

$$\int_1^{\infty}f(x)\,dx,$$

which therefore *must* converge.

c. Since $0<f(x)<g(x)$ for $x\geqslant 1$, the series

$$\sum_{n=1}^{\infty}g(n)$$

might converge. For example, if $f(x)=1/x^2$ and $g(x)=2/x^2$ then $\sum g(n)$ converges, but if $f(x)=1/x^2$ and $g(x)=2/x$ the $\sum g(n)$ diverges. then

d. Since the series $\sum f(n)$ converges, $\lim f(n) = 0$ by the Vanishing Criterion, so $\lim\{1/f(n)\} \neq 0$. Therefore the series $\sum\{1/f(n)\}$ *cannot* converge.

e. Since it is given that the series $\sum f(n)$ converges, and that $f(n)$ and $g(n)$ are positive for $n \geqslant 1$, if $\lim\{g(n)/f(n)\} = 5^8$ the series $\sum g(n)$ *must* converge by the Limit Comparison Test.

12. a. Since
$$\frac{\arctan n}{\sqrt{n+1}} \geqslant \frac{\pi/4}{\sqrt{n+n}} = \frac{\pi}{4\sqrt{2}} \cdot \frac{1}{\sqrt{n}} > 0$$
for $n \geqslant 1$, and since $\sum n^{-1/2}$ is a divergent p-series ($p = \frac{1}{2} > 1$), the series
$$\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n+1}}$$
diverges by the Comparison Test.

b. Since
$$\lim\left\{\frac{n^2+1}{(n+1)^2}\cos\left(\frac{\pi}{4n}\right)\right\} = \lim\left\{\frac{1+1/n^2}{(1+1/n)^2}\cos\left(\frac{\pi}{4n}\right)\right\} = 1\cdot\cos 0 = 1,$$
the series
$$\sum_{n=1}^{\infty} \frac{n^2+1}{(n+1)^2}\cos\left(\frac{\pi}{4n}\right)$$
diverges by the Vanishing Criterion.

c. Since
$$\frac{3k}{k^3+1} < \frac{3}{k^2}$$
for $k \geqslant 1$, and $\sum k^{-2}$ is a convergent p-series ($p = 2 > 1$), the series
$$\sum_{k=1}^{\infty} \frac{3k}{k^3+1}$$
converges by the Comparison Test. Also, the series
$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k}$$
is a convergent geometric series ($|r| = \frac{2}{3} < 1$), and therefore the series
$$\sum_{k=1}^{\infty}\left(\frac{3k}{k^3+1} - \frac{2^{k-1}}{3^k}\right)$$
is convergent (being a difference of convergent series).

13. a. Let $a_n = \big(2^n n^2\big)/\big((2n)!\big)$; then
$$\begin{aligned}\lim \frac{a_{n+1}}{a_n} &= \lim\left\{\frac{2^{n+1}(n+1)^2}{(2n+2)!}\cdot\frac{(2n)!}{2^n n^2}\right\}\\ &= \lim \frac{n+1}{n^2(2n+1)}\\ &= \lim\left\{\frac{1}{n^2}\cdot\frac{1+1/n}{(2+1/n)}\right\}\\ &= 0,\end{aligned}$$
and $0 < 1$, so the series $\sum(-1)^{n+1}a_n$ is absolutely convergent by the Ratio Test.

b. Let $a_n = n^{-1/3}$, and observe that $\cos\big((n+1)\pi\big)a_n = (-1)^{n+1}a_n$. Since $\sum a_n$ is a divergent p-series ($p = \frac{1}{3} \leqslant 1$) the series $\sum(-1)^{n+1}a_n$ is not absolutely convergent. On the other hand, a_n is positive and decreasing (since it is the reciprocal of an increasing function) for $n \geqslant 1$, and $\lim a_n = 0$, so $\sum(-1)^{n+1}a_n$ is convergent by the Alternating Series Test. Therefore, the series $\sum(-1)^{n+1}a_n$ is conditionally convergent.

c. Let
$$a_n = (-4)^n\left(2 - \frac{1}{n}\right)^{2n}.$$
Since $|a_n| \geqslant 4^n$ for $n \geqslant 1$, it follows that $\lim a_n \neq 0$. Therefore, $\sum a_n$ diverges by the Vanishing Criterion.

d. Since
$$0 < \left|\frac{\sin(2^n)}{(n+1)(n+2)}\right| < \frac{1}{n^2}$$
for $n \geqslant 1$, and $\sum n^{-2}$ is a convergent p-series ($p = 2 > 1$), the Comparison Test implies that the series
$$\sum_{n=1}^{\infty}\left|\frac{\sin(2^n)}{(n+1)(n+2)}\right|$$
is convergent. Therefore, the series in question is absolutely convergent.

14. Let
$$\alpha_n = (-1)^n\frac{(x-2)^n}{2^n n};$$
then
$$\begin{aligned}\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| &= \lim\left|\frac{(x-2)^{n+1}}{2^{n+1}(n+1)}\cdot\frac{2^n n}{(x-2)^n}\right|\\ &= \tfrac{1}{2}|x-2|\lim\frac{1}{1+1/n}\\ &= \tfrac{1}{2}|x-2|.\end{aligned}$$
So $\sum \alpha_n$ converges absolutely if $\frac{1}{2}|x-2| < 1$, *i.e.*$0 < x < 4$, by the Ratio Test. If $x = 0$ then $\sum\alpha_n = \sum 1/n$ is a divergent p-series ($p = 1 \leqslant 1$), so 0 does not belong to the interval of convergence. If $x = 4$ then $\alpha_n = (-1)^n/n$. Since $1/n$ is positive and decreasing (because it is the reciprocal of an increasing function) for $n \geqslant 1$, and $\lim\{1/n\} = 0$, the series $\sum(-1)^n/n$ converges by the Alternating Series Test. So 4 does belong to the interval of convergence. Therefore, the interval of convergence of the power series $\sum\alpha_n$ is $(\,0, 4\,]$.

15. Using the Maclaurin series of the exponential function gives
$$e^{-2x} = \sum_{k=0}^{\infty}\frac{(-2x)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k 2^k}{k!}x^k.$$

16. Since (using the binomial series)
$$\begin{aligned}(1+t)^{-2} &= 1 + \sum_{k=1}^{\infty}\frac{(-2)(-3)\cdots(-2-k+1)}{k!}t^k\\ &= \sum_{k=0}^{\infty}(-1)^k(k+1)t^k,\end{aligned}$$
for $-1 < t < 1$, it follows that it follows that
$$\begin{aligned}\frac{1}{(2x-1)^2} &= \big(1 + 2(x-1)\big)^{-2}\\ &= \sum_{k=0}^{\infty}(-1)^k\big(k+1\big)\big(2(x-1)\big)^k\\ &= \sum_{k=0}^{\infty}(-1)^k 2^k(k+1)(x-1)^k\\ &= 1 - 4(x-1) + 12(x-1)^2 - 32(x-1)^3 + 80(x-1)^4 - \cdots,\end{aligned}$$
for $-1 < 2(x-1) < 1$, *i.e.*, $\frac{1}{2} < x < \frac{3}{2}$.