1. Evaluate each of the following limits.

a. $\lim_{x\to\pi} \dfrac{\sin^2 x}{1+\cos 3x}$ b. $\lim_{x\to\infty} x^2 \log\left(1+\dfrac{4}{x^2}\right)$

2. Evaluate each of the following integrals.

a. $\displaystyle\int_1^5 \frac{x+2}{\sqrt{2x-1}}\,dx$ b. $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-4}}$ c. $\displaystyle\int \frac{\arctan x}{x^2}\,dx$

d. $\displaystyle\int \frac{\sec^4\sqrt{x}\,\tan^2\sqrt{x}}{\sqrt{x}}\,dx$ e. $\displaystyle\int_0^{\frac{1}{2}} \frac{x+\arccos x}{\sqrt{1-x^2}}\,dx$

f. $\displaystyle\int \frac{e^x\,dx}{\sqrt{3-2e^x-e^{2x}}}$ g. $\displaystyle\int \frac{3x^2-2}{x^2-2x-8}\,dx$

3. Find the area or the region between the graphs of $y=\sqrt{x-1}$ and $y=x-1$ on the interval $[\,1,5\,]$.

4. Let $\mathscr{R}$ be the region bounded by the graphs of $y=2e^x$ and $y=2\log x$, between $x=1$ and $x=e$.

a. Find the volume of the solid obtained by revolving $\mathscr{R}$ about the y-axis.

b. Find the volume of the solid obtained by revolving $\mathscr{R}$ about the line $y=-1$.

5. Explain why the integral is improper and evaluate it or show that it diverges.

a. $\displaystyle\int_{\frac{1}{2}}^{\infty} \frac{\arctan(2x)}{4x^2+1}\,dx$ b. $\displaystyle\int_0^6 \frac{dx}{(x-2)^3}$

6. Solve the differential equation

$$\frac{dy}{dx} = 1+x^2+y^2+x^2y^2$$

with $y(0)=1$. Express y as a function of x.

7. Find the sum of the series

$$\sum_{k=1}^{\infty} \log\left(\frac{8k^2-2k-1}{8k^2-2k-3}\right).$$

8. Find the limit of the sequence $\{a_n\}$, where

a. $a_n = \dfrac{\sin(n!)}{n+1}$ b. $a_n = \dfrac{(-1)^n 3^{n+2}}{2^{2n+1}}$

9. Provide an example of a geometric series whose sum is $\frac{1}{12}\pi$.

10. Determine whether each of the following series is convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{2}{3n^2}\right)^n$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{\arctan n}}{n^2+1}$

11. Determine whether each series is absolutely convergent, conditionally convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\sqrt[3]{n^3+1}}{1+n+n^3}$ b. $\displaystyle\sum_{n=2}^{\infty}\frac{(-2)^n}{(\log n)^n}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$

12. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{(x+1)^n}{5^n\sqrt[3]{n+1}}.$$

13. Find the Maclaurin series of $\log(x+2)$, and its interval of convergence.

1. a. Using the Maclaurin expansions of $\sin^2 t = \frac{1}{2}(1-\cos 2t)$ and $1-\cos 3t$, where $t = \pi - x$, gives

$$\lim_{x\to\pi} \frac{\sin^2 x}{1+\cos 3x} = \frac{1}{2}\lim_{t\to 0} \frac{2-\frac{2}{3}t^2+\frac{4}{45}t^4-\cdots}{\frac{9}{2}-\frac{27}{8}t^2+\frac{81}{80}t^4-\cdots} = \frac{2}{9}.$$

b. Using the Maclaurin expansion of $\log(1+4t)$, where $t = 1/x^2$, yields

$$\lim_{x\to\infty} x^2\log\left(1+\frac{4}{x^2}\right) = \lim_{t\to 0^+}\left\{4-8t+\tfrac{64}{3}t^2-\cdots\right\} = 4.$$

2. a. Partial integration gives

$$\int_1^5 \frac{x+2}{\sqrt{2x-1}}\,dx = (x+2)\sqrt{2x-1}\,\Big|_1^5 - \int_1^5 \sqrt{2x-1}\,dx$$
$$= 18 - \tfrac{1}{3}(2x-1)^{3/2}\,\Big|_1^5 = 18 - \tfrac{26}{3}$$
$$= \tfrac{28}{3}.$$

b. Subtracting and adding $\frac{1}{4}x^2$, and then integrating by parts, gives

$$\int \frac{dx}{x^3\sqrt{x^2-4}} = -\tfrac{1}{4}\int \frac{\sqrt{x^2-4}}{x^3} + \tfrac{1}{4}\int \frac{dx}{x\sqrt{x^2-4}}$$
$$= \frac{\sqrt{x^2-4}}{8x^2} + \tfrac{1}{8}\int \frac{dx}{x\sqrt{x^2-4}}$$
$$= \frac{\sqrt{x^2-4}}{8x^2} + \tfrac{1}{16}\operatorname{arcsec}\tfrac{1}{2}x + C.$$

c. Partial integration gives

$$\int \frac{\arctan x}{x^2}\,dx = -\frac{\arctan x}{x^2} + \int \frac{dx}{x(x^2+1)}.$$

In the remaining integral, let $t = 1+1/x^2$, so that $dx/x^3 = -\frac{1}{2}\,dt$, and hence

$$\int \frac{dx}{x(x^2+1)}\,dx = \int \frac{1}{1+1/x^2}\cdot\frac{dx}{x^3} = -\tfrac{1}{2}\log(1+1/x^2) + C.$$

Therefore,

$$\int \frac{\arctan x}{x^2}\,dx = -\frac{\arctan x}{x^2} + \tfrac{1}{2}\log\frac{x^2}{x^2+1} + C.$$

d. Let $t = \tan\sqrt{x}$, so that $2\,dt = \sec^2\sqrt{x}\,dx/\sqrt{x}$, $\sec^2\sqrt{x} = t^2+1$, and hence

$$\int \frac{\sec^4\sqrt{x}\,\tan^2\sqrt{x}}{\sqrt{x}}\,dx = 2\int (t^2+1)t^2\,dt = 2\int (t^4+t^2)\,dt$$
$$= \tfrac{2}{5}\tan^5\sqrt{x} + \tfrac{2}{3}\tan^3\sqrt{x} + C.$$

e. Separating the integrand and integrating each term by inspection gives

$$\int_0^{\frac{1}{2}} \frac{x+\arccos x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}\,\Big|_0^{\frac{1}{2}} - \tfrac{1}{2}(\arccos x)^2\,\Big|_0^{\frac{1}{2}}$$
$$= 1 - \tfrac{1}{2}\sqrt{3} + \tfrac{5}{72}\pi^2.$$

f. Let $t = e^x+1$, then $dt = e^x\,dx$, $3-2e^x-e^{2x} = 4-(e^x+1)^2 = 4-t^2$, and therefore

$$\int \frac{e^x\,dx}{\sqrt{3-2e^x-e^{2x}}} = \int \frac{dt}{\sqrt{4-t^2}} = \arcsin\tfrac{1}{2}(e^x+1) + C.$$

g. Dividing and then resolving the proper rational function into partial fractions gives

$$\frac{3x^2-2}{x^2-2x-8} = 3 + \frac{6x+22}{(x-4)(x+2)} = 3 + \frac{23}{3(x-4)} - \frac{5}{3(x+2)};$$

therefore,

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = 3x + \tfrac{1}{3}\log\left|\frac{(x-4)^{23}}{(x+2)^5}\right| + C.$$

3. By inspection (or by solving the equation $\sqrt{x-1} = x-1$), one finds that the graph of $y = x-1$ meets the graph of $y = \sqrt{x-1}$ at the points $(1,0)$ and $(2,1)$. On $[1,2]$, $\sqrt{x-1} \geqslant x-1$, and the area between the curves is equal to

$$\int_1^2 (\sqrt{x-1}-x+1)\,dx = \left\{\tfrac{2}{3}(x-1)^{3/2} - \tfrac{1}{2}x^2 + x\right\}\Big|_1^2 = \tfrac{1}{6}.$$

On $[2,5]$, $\sqrt{x-1} \leqslant x-1$, and the area between the curves is equal to

$$\int_2^5 (x-1-\sqrt{x-1})\,dx = \left\{\tfrac{1}{2}x^2 - x - \tfrac{2}{3}(x-1)^{3/2}\right\}\Big|_2^5 = \tfrac{17}{6}.$$

Therefore, the area between the curves on $[1,5]$ is equal to $\frac{1}{6}+\frac{17}{6} = 3$.

4. a. The solid in question can be decomposed into cylindrical shells of radius x and height $2(e^x-\log x)$, for $1 \leqslant x \leqslant e$, and so its volume is equal to

$$4\pi\int_1^e x(e^x-\log x)\,dx.$$

Since (by partial integration)

$$\int_1^e xe^x\,dx = (x-1)e^x\,\Big|_1^e = e^e(e-1),$$

and (also by partial integration)

$$\int_1^e x\log x\,dx = \tfrac{1}{4}x^2(2\log x-1)\,\Big|_1^e = \tfrac{1}{4}(e^2+1),$$

it follows that the volume of the solid in question is equal to

$$\pi\big(4e^e(e-1)-e^2-1\big).$$

b. The solid in question can be decomposed into annuli of inner radius $2\log x+1$ and outer radius $2e^x+1$, for $1 \leqslant x \leqslant e$, and so its volume is equal to

$$\pi\int_1^e \big((2e^x+1)^2-(2\log x+1)^2\big)\,dx.$$

Now,

$$\int_1^e (2e^x+1)^2\,dx = \int_1^e (4e^{2x}+4e^x+1)\,dx$$
$$= (2e^{2x}+4e^x+x)\,\Big|_1^e$$
$$= 2e^{2e}+4e^e-2e^2-3e-1,$$

and (by partial integration)

$$\int_1^e (2\log x+1)^2\,dx = \int_1^e \big(4(\log x)^2+4\log x+1\big)\,dx$$
$$= 4x(\log x)^2\,\Big|_1^e - 4\int_1^e \log x\,dx + \int_1^e dx$$
$$= 4e - (4x\log x)\,\Big|_1^e + 5\int_1^e dx$$
$$= 5(e-1).$$

Therefore, the volume of the solid in question is equal to

$$2\pi(e^{2e}+2e^e-e^2-4e+2).$$

5. a. One has

$$\int_{\frac{1}{2}}^{\infty} \frac{\arctan 2x}{1+4x^2}\,dx = \lim_{\alpha\to\infty}\tfrac{1}{4}(\arctan 2x)^2\,\Big|_{\frac{1}{2}}^{\alpha} = \tfrac{3}{64}\pi^2,$$

where an antiderivative of the integrand can be found by inspection (or by changing the variable of integration to $t = \arctan 2x$).

b. Since

$$\int_0^2 \frac{dx}{(x-2)^3} = \lim_{\alpha\to 2^-}\frac{-1}{2(x-2)^2}\,\Big|_0^{\alpha} = \tfrac{1}{8} + \lim_{\alpha\to 2^-}\frac{-1}{2(\alpha-2)^2} = -\infty,$$

and

$$\int_2^6 \frac{dx}{(x-2)^3} = \lim_{\beta\to 2^+}\frac{-1}{2(x-2)^2}\,\Big|_{\beta}^{6} = -\tfrac{1}{32} + \lim_{\beta\to 2^+}\frac{1}{2(\beta-2)^2} = \infty,$$

the integral in question is divergent (but *not* to ∞, *nor* to $-\infty$).

6. Factoring the right hand side of the given differential equation yields

$$\frac{dy}{dx} = (x^2+1)(y^2+1), \qquad \text{or} \qquad \frac{1}{y^2+1}\frac{dy}{dx} = x^2+1,$$

and therefore,

$$3\arctan y = x^3+3x+C.$$

The initial condition $y(0) = 1$ implies that $\frac{3}{4}\pi = C$, so the explicit solution of the initial value problem is

$$y = \tan\tfrac{1}{12}(4x^3+12x+3\pi).$$

7. Notice that

$$a_k = \log\frac{(2k-1)(4k+1)}{(4k-3)(2k+1)} = b_k - b_{k+1}, \quad \text{where} \quad b_k = \log\frac{2k-1}{4k-3},$$

and so $a_1 + \cdots + a_k = b_1 - b_{k+1}$, which implies that

$$\sum_{k=1}^{\infty} \log\frac{(2k-1)(4k+1)}{(4k-3)(2k+1)} = b_1 - \lim b_{k+1} = \log 1 - \log\tfrac{1}{2} = \log 2.$$

8. a. Since $|\sin(n!)/(n+1)| < 1/n$, and $1/n \to 0$, the sequence in question plainly converges to zero.

b. The sequence in question converges to zero by the Vanishing Criterion, since it is the sequence of terms of a convergent geometric series ($r = -\frac{3}{4}$; $|r| < 1$).

9. $\frac{1}{12}\pi$ (*i.e.*, take $r = 0$). Alternatively (and less trivially), $\displaystyle\sum_{n=1}^{\infty}\frac{\pi}{13^n}$ will do.

10. a. Since

$$\left(1 - \frac{2}{3n^2}\right)^n \geqslant \left(1 - \frac{2}{3n}\right)^n$$

for $n \geqslant 1$, and

$$\lim\left(1 - \frac{2}{3n}\right)^n = e^{-2/3},$$

the series

$$\sum_{n=1}^{\infty}\left(1 - \frac{2}{3n^2}\right)^n$$

diverges by the Vanishing Criterion.

b. Let $a_n = 2^{n^2}/n!$; then

$$\lim\frac{a_{n+1}}{a_n} = \lim\frac{2^{(n+1)^2}}{(n+1)!}\cdot\frac{n!}{2^{n^2}} = \lim\frac{2^{2n+1}}{n+1} = \infty,$$

by elementary properties of the logarithm, the series $\sum a_n$ diverges by the Ratio Test.

c. Since

$$0 < \frac{\sqrt{\arctan n}}{n^2+1} < \frac{\sqrt{\frac{1}{2}\pi}}{n^2},$$

for $n \geqslant 1$, the series

$$\sum_{n=1}^{\infty}\frac{\sqrt{\arctan n}}{n^2+1}$$

converges with the p-series $\sum n^{-2}$ ($p = 2 > 1$), by the Comparison Test.

11. a. If

$$a_n = \frac{(-1)^n\sqrt[3]{n^3+1}}{n^3+n+1},$$

then $0 < |a_n| < n^{-2}$ for $n \geqslant 1$, and so the series $\sum|a_n|$ converges with the p-series $\sum n^{-2}$ ($p = 2 > 1$), by the Comparison Test. Therefore, the series $\sum a_n$ is absolutely convergent.

b. If

$$a_n = \frac{(-2)^n}{(\log n)^n},$$

then $\lim\sqrt[n]{|a_n|} = \lim\{2/(\log n)\} = 0$, so $\sum a_n$ is absolutely convergent by the Root Test.

c. The series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+3}} = -\sum_{k=4}^{\infty}\frac{(-1)^k}{\sqrt{k}},$$

is (a non-zero multiple of) a conditionally convergent alternating p-series ($p = \frac{1}{2}$; $0 < p \leqslant 1$), less three terms.

12. Let

$$\alpha_n = \frac{(x+1)^n}{5^n\sqrt[3]{n+1}};$$

then

$$\begin{aligned}\lim\left|\frac{\alpha_{n+1}}{\alpha_n}\right| &= \lim\left|\frac{(x+1)^{n+1}}{5^{n+1}\sqrt[3]{n+2}}\cdot\frac{5^n\sqrt[3]{n+1}}{(x+1)^n}\right| \\ &= \tfrac{1}{5}|x+1|\lim\sqrt[3]{\frac{1+1/n}{1+2/n}} \\ &= \tfrac{1}{5}|x+1|.\end{aligned}$$

So $\sum\alpha_n$ is absolutely convergent if $\frac{1}{5}|x+1| < 1$, *i.e.*, $-6 < x < 4$, and the radius of convergence of $\sum\alpha_n$ is 5. If $x = -6$ then $\alpha_n = (-1)^n\sqrt[3]{(n+1)^{-1}}$ and if $x = 4$ then $\alpha_n = \sqrt[3]{(n+1)^{-1}}$. Now

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt[3]{n+1}} = -\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt[3]{k}}$$

is (a non-zero multiple of) a conditionally convergent alternating p series ($p = \frac{1}{3}$; $0 < p \leqslant 1$), so the series $\sum\alpha_n$ converges if $x = -6$ and diverges if $x = 4$. Therefore, the interval of convergence of $\sum\alpha_n$ is $[-6, 4)$.

13. Since

$$\log(1+t) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}t^k,$$

for $-1 < t \leqslant 1$, it follows that

$$\begin{aligned}\log(x+2) &= \log 2 + \log(1 + \tfrac{1}{2}x) \\ &= \log 2 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(\tfrac{1}{2}x)^k \\ &= \log 2 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2^k k}x^k,\end{aligned}$$

for $-1 < \frac{1}{2}x \leqslant 1$, *i.e.*, $-2 < x \leqslant 2$; so the interval of convergence of the Maclaurin series in question is $(-2, 2]$.