1. Given
$$f(x) = \arctan\left(\frac{1}{x+1}\right),$$
find $f'(x)$ and simplify your answer.

2. Evaluate each of the following integrals.

a. $\displaystyle\int \frac{x^2}{\sqrt{x-4}}\,dx$ b. $\displaystyle\int \frac{x\arcsin(x^2)}{\sqrt{1-x^4}}\,dx$ c. $\displaystyle\int_0^{\frac{1}{4}\pi} \sqrt{\tan x}\,\sec^4 x\,dx$

d. $\displaystyle\int (\cos^2\vartheta + \sin^3\vartheta)\,d\vartheta$ e. $\displaystyle\int \frac{\sqrt{9x^2-4}}{x}\,dx$

f. $\displaystyle\int \frac{6x^2-5x-1}{(x-2)(x^2+9)}\,dx$ g. $\displaystyle\int 16x\arctan 4x\,dx$

3. Evaluate each of the following limits, using ∞ or $-\infty$ as appropriate. Justify your answers.

a. $\displaystyle\lim_{x\to 0^+} \tan x\,\ln x$ b. $\displaystyle\lim_{x\to 3^-}\left\{\frac{1}{\ln(x-2)} - \frac{1}{x-3}\right\}$

c. $\displaystyle\lim_{x\to\infty} (e^{2x}+e^{-2x})^{1/x}$

4. Evaluate each of the following improper integrals.

a. $\displaystyle\int_6^\infty \frac{dx}{x\sqrt{x^2-9}}$ b. $\displaystyle\int_0^{10} \frac{dx}{(x-4)^2}$

5. Let $\mathscr{R}$ be the region bounded by the graphs of $y = \sin x$ and $y = \sqrt{3}\cos x$ from $x = 0$ to $x = \frac{1}{2}\pi$. Sketch the region $\mathscr{R}$ and find its area.

6. Give the explicit solution of the differential equation
$$x - 8y\sqrt{x^2+1}\,\frac{dy}{dx} = 0$$
which satisfies the initial condition $y(0) = 1$.

7. Let $\mathscr{R}$ be the region bounded by the graphs of
$$y = \frac{1}{x+1}, \qquad y = -x^2, \qquad x = 0 \qquad \text{and} \qquad x = 2.$$

a. Find the volume of the solid obtained by revolving $\mathscr{R}$ about the y-axis.

b. Compute the volume of the solid obtained by revolving $\mathscr{R}$ about the line $y = 3$.

8. Does the sequence $\{n^3/n!\}$ converge? If so, find its limit. If not explain why not.

9. The sequence $\{a_n\}_{n\geqslant 1}$ is defined by
$$a_1 = 1, \qquad \text{and} \qquad a_{n+1} = 3 - \frac{1}{a_n} \qquad \text{for} \qquad n \geqslant 1.$$

a. Find the first four terms of $\{a_n\}_{n\geqslant 1}$.

b. Show that the sequence $\{a_n\}$ is increasing and bounded.

c. What theorem implies that the sequence $\{a_n\}$ converges?

d. Find $\lim a_n$.

10. Determine whether the series converges or diverges. If a series converges, find its sum.

a. $\displaystyle\sum_{n=2}^\infty \frac{(-2)^{n-1}}{3^{n+1}}$ b. $\displaystyle\sum_{n=1}^\infty \frac{2}{n(n+2)}$

11. Investigate the convergence of each series. Justify your assertions carefully.

a. $\displaystyle\sum_{n=1}^\infty \frac{-n^5+n^2+1}{n^5+2}$ b. $\displaystyle\sum_{n=2}^\infty \frac{1}{n\sqrt{\ln n}}$

c. $\displaystyle\sum_{k=0}^\infty \frac{k}{4k^3-5}$ d. $\displaystyle\sum_{k=1}^\infty \frac{k!}{k^k}$

12. Determine whether each series is absolutely convergent, conditionally convergent or divergent. Justify your assertions carefully.

a. $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{(\arctan n)^n}$ b. $\displaystyle\sum_{m=1}^\infty \frac{\sin m}{m^2+m}$ c. $\displaystyle\sum_{k=1}^\infty (-1)^k\frac{k^2+1}{k^3}$

13. You are given that $\{a_n\}_{n\geqslant 1}$ is a decreasing sequence of positive real numbers such that $\lim\{na_n\} = 0$.

a. Show that $\sum_{n=1}^\infty (-1)^{n-1}a_n$ converges.

b. Give an example to show that $\sum_{n=1}^\infty a_n$ need not converge.

14. Find the radius and interval of convergence of the power series
$$\sum_{n=1}^\infty \frac{e^n}{n^3}(x-4)^n.$$

15. Find the Maclaurin series of
$$f(x) = \frac{1}{(1+x)^3},$$
and write the first four non-zero terms of the series explicitly.

1. Using the derivative of the inverse tangent function and the Chain Rule, one obtains

$$f'(x) = \frac{1}{1+\big(1/(x+1)\big)^2}\cdot\frac{-1}{(x+1)^2} = \frac{-1}{(x+1)^2+1}, \qquad \text{or} \qquad \frac{-1}{x^2+2x+2}.$$

2. a. Repeated partial integration (integrating the rational power and differentiating the polynomial) gives

$$\begin{aligned}\int \frac{x^2}{\sqrt{x-4}}\,dx &= 2x^2\sqrt{x-4} - \tfrac{8}{3}x(x-4)^{3/2} + \tfrac{16}{15}(x-4)^{5/2} + C\\ &= \tfrac{2}{15}\big(15x^2 - 20x(x-4) + 8(x-4)^2\big)\sqrt{x-4} + C\\ &= \tfrac{2}{15}(3x^2+16x+128)\sqrt{x-4} + C.\end{aligned}$$

b. If $t = \arcsin(x^2)$ then $\frac{1}{2}\,dt = x\arcsin(x^2)/\sqrt{1-x^4}\,dx$, and so

$$\int \frac{x\arcsin(x^2)}{\sqrt{1-x^4}}\,dx = \tfrac{1}{2}\int t\,dt = \tfrac{1}{4}t^2 + C = \tfrac{1}{4}\big(\arcsin(x^2)\big)^2 + C.$$

c. If $t = \tan x$ then $dt = \sec^2 x\,dx$ and $\sec^2 x = t^2+1$; also, $t=0$ if $x=0$ and $t=1$ if $x = \frac{1}{4}\pi$. This gives

$$\begin{aligned}\int_0^{\frac{1}{4}\pi} \sqrt{\tan x}\,\sec^4 x\,dx &= \int_0^1 t^{1/2}(t^2+1)\,dt = \int_0^1 (t^{5/2}+t^{1/2})\,dt\\ &= \big(\tfrac{2}{7}t^{7/2} + \tfrac{2}{3}t^{3/2}\big)\Big|_0^1\\ &= \tfrac{2}{7} + \tfrac{2}{3} = \tfrac{20}{21}.\end{aligned}$$

d. Writing the integral in question as the sum of two integrals, and then applying the half angle identity for the cosine to the first integral and the change of variables $t = \cos\vartheta$ (so $-dt = \sin\vartheta\,d\vartheta$ and $\sin^2\vartheta = 1-t^2$) to the second integral, gives

$$\begin{aligned}\int(\cos^2\vartheta + \sin^3\vartheta)\,d\vartheta &= \tfrac{1}{2}\int(1+\cos 2\vartheta)\,d\vartheta + \int(t^2-1)\,dt\\ &= \tfrac{1}{2}\vartheta + \tfrac{1}{4}\sin 2\vartheta + \tfrac{1}{3}t^3 - t + C\\ &= \tfrac{1}{2}\vartheta + \tfrac{1}{4}\sin 2\vartheta + \tfrac{1}{3}\cos^3\vartheta - \cos\vartheta + C.\end{aligned}$$

e. If $3x = 2\sec\vartheta$ then $dx = \frac{2}{3}\sec\vartheta\tan\vartheta\,d\vartheta$ and $\sqrt{9x^2-4} = 2\tan\vartheta$, which gives

$$\begin{aligned}\int \frac{\sqrt{9x^2-4}}{x}\,dx &= \int \frac{(2\tan\vartheta)(\frac{2}{3}\sec\vartheta\tan\vartheta)}{\frac{2}{3}\sec\vartheta}\,d\vartheta\\ &= 2\int \tan^2\vartheta\,d\vartheta = 2\int(\sec^2 -1)\,d\vartheta\\ &= 2\tan\vartheta - 2\vartheta + C\\ &= \sqrt{9x^2-4} - 2\,\mathrm{arcsec}\big(\tfrac{3}{2}x\big) + C.\end{aligned}$$

f. Resolving the integrand into partial fractions and integrating term-by-term gives

$$\begin{aligned}\int \frac{6x^2-5x-1}{(x-2)(x^2+9)}\,dx &= \int\left\{\frac{1}{x-2} + \frac{5x+5}{x^2+9}\right\}dx\\ &= \ln|x-2| + \tfrac{5}{2}\ln(x^2+9) + \tfrac{5}{3}\arctan\big(\tfrac{1}{3}x\big) + C.\end{aligned}$$

g. Partial integration (integrating the power and differentiating the inverse tangent) gives

$$\begin{aligned}\int 16x\arctan 4x\,dx &= \tfrac{1}{2}(16x^2+1)\arctan 4x - 2\int\frac{16x^2+1}{1+16x^2}\,dx\\ &= \tfrac{1}{2}(16x^2+1)\arctan 4x - 2x + C,\end{aligned}$$

where the antiderivative $\frac{1}{2}(16x^2+1)$ of $16x$ was chosen for the simplicity of the remaining integral.

3. a. Revising the expression in the limit, before and after applying l'Hôpital's Rule, gives

$$\begin{aligned}\lim_{x\to 0^+}\frac{\log x}{\cot x} &= \lim_{x\to 0^+}\frac{1/x}{\csc^2 x} = \lim_{x\to 0^+}\left(\frac{\sin x}{x}\cdot\sin x\right)\\ &= 1\cdot 0 = 0.\end{aligned}$$

b. Expanding $\log(x-2)$ about 3 and simplifying gives

$$\begin{aligned}\lim_{x\to 3^-}\left\{\frac{1}{\log(x-2)} - \frac{1}{x-3}\right\} &= \lim_{x\to 3^-}\frac{(x-3)-\log\big(1-(x-3)\big)}{(x-3)\log\big(1+(x-3)\big)}\\ &= \lim_{x\to 3^-}\frac{\frac{1}{2} - \frac{1}{3}(x-3) + \frac{1}{4}(x-3)^2 - \cdots}{1 - \frac{1}{2}(x-3) + \frac{1}{3}(x-3)^2 - \cdots}\\ &= \tfrac{1}{2}.\end{aligned}$$

c. Applying l'Hôpital's Rule and extracting dominant terms gives

$$\begin{aligned}\lim_{x\to\infty}\frac{\log(e^{2x}+e^{-2x})}{x} &= \lim_{x\to\infty}\frac{2\big(e^{2x}-e^{-2x}\big)}{e^{2x}+e^{-2x}}\\ &= \lim_{x\to\infty}\frac{2\big(1-e^{-4x}\big)}{1+e^{-4x}}\\ &= 2;\end{aligned}$$

and therefore

$$\lim_{x\to\infty}(e^{2x}+e^{-2x})^{1/x} = e^2.$$

4. a. A standard integral formula gives

$$\begin{aligned}\int_6^\infty \frac{dx}{x\sqrt{x^2-9}} &= \lim_{\alpha\to\infty}\tfrac{1}{3}\,\mathrm{arcsec}\big(\tfrac{1}{3}x\big)\Big|_6^\alpha\\ &= \lim_{\alpha\to\infty}\tfrac{1}{3}\big(\mathrm{arcsec}\big(\tfrac{1}{3}\alpha\big) - \mathrm{arcsec}\,2\big)\\ &= \tfrac{1}{3}\big(\tfrac{1}{2}\pi - \tfrac{1}{3}\pi\big) = \tfrac{1}{18}\pi.\end{aligned}$$

b. Since

$$\int_0^4 \frac{dx}{(x-4)^2} = \lim_{\alpha\to 4^-}\frac{-1}{x-4}\Big|_0^\alpha = \tfrac{1}{4} + \lim_{\alpha\to 4^-}\frac{-1}{\alpha-4} = \infty$$

and

$$\int_4^{10} \frac{dx}{(x-4)^2} = \lim_{\beta\to 4^+}\frac{-1}{x-4}\Big|_\beta^{10} = -\tfrac{1}{6} + \lim_{\beta\to 4^+}\frac{1}{\beta-4} = \infty,$$

the integral in question diverges to ∞.

5. The curves meet where $\sin x = \sqrt{3}\cos x$, *i.e.*, where $\tan x = \sqrt{3}$, or $x = \frac{1}{3}\pi + k\pi$, where k is an integer. So the only point of intersection over the interval $[\,0, \frac{1}{2}\pi\,]$ is $(\frac{1}{3}\pi, \frac{1}{2}\sqrt{3})$. On the interval $(\,0, \frac{1}{3}\pi\,)$, $\sin x < \sqrt{3}\cos x$ and on the interval $(\,\frac{1}{3}\pi, \frac{1}{2}\pi\,)$ $\sin x > \sqrt{3}\cos x$, so the area of $\mathscr{R}$ is

$$\begin{aligned}&\int_0^{\frac{1}{3}\pi}\big(\sqrt{3}\cos x - \sin x\big)\,dx + \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\big(\sin x - \sqrt{3}\cos x\big)\,dx\\ &\qquad = \big(\sqrt{3}\sin x + \cos x\big)\Big|_0^{\frac{1}{3}\pi} + \big(-\cos x - \sqrt{3}\sin x\big)\Big|_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\\ &\qquad = \big(2-1\big) + \big(-\sqrt{3}+2\big) = 3-\sqrt{3}.\end{aligned}$$

6. Separating variables and integrating gives

$$\int 8y\,dy = \int\frac{x}{\sqrt{x^2+1}}\,dx, \qquad \text{or} \qquad 4y^2 = \sqrt{x^2+1} + C,$$

and the initial condition $y(0) = 1$ implies that $4 = 1 + C$, or $C = 3$. Therefore, $4y^2 = \sqrt{x^2+1} + 3$, or $y = \frac{1}{2}\sqrt{\sqrt{x^2+1}+3}$.

7. On the interval $[\,0, 2\,]$, $-x^2 < 1/(x+1)$; *i.e.*, the curves do not meet and $\mathscr{R}$ is the region below the graph of $y = 1/(x+1)$, above the graph of $y = -x^2$, to the right of the y-axis and to the left of the line defined by $x = 2$.

a. The solid in question can be decomposed into cylindrical shells of radius x and height $1/(x+1) + x^2$, for $0 \leqslant x \leqslant 2$, so its volume is

$$\begin{aligned}2\pi\int_0^2\left\{\frac{x}{x+1} + x^3\right\}dx &= 2\pi\big\{x - \log(x+1) + \tfrac{1}{4}x^4\big\}\Big|_0^2\\ &= 2\pi(6-\log 3).\end{aligned}$$

b. Cross sections perpendicular to the x-axis of the solid in question are annuli of inner radius $3 - 1/(x+1)$ and outer radius $3 + x^2$, for $0 \leqslant x \leqslant 2$, so the volume of the solid is

$$\begin{aligned}\pi \int_0^2 \left\{ (3+x^2)^2 - \left(3 - \frac{1}{x+1}\right)^2 \right\} dx \\ &= \pi \int_0^2 \left\{ x^4 + 6x^2 + \frac{6}{x+1} - \frac{1}{(x+1)^2} \right\} dx \\ &= \pi \left\{ \tfrac{1}{5}x^5 + 2x^3 + 6\log(x+1) + \frac{1}{x+1} \right\} \Big|_0^2 \\ &= \tfrac{326}{15}\pi + 6\pi \log 3 = \tfrac{2}{15}\pi(163 + 45\log 3).\end{aligned}$$

8. Since

$$\begin{aligned} 0 < \frac{n^3}{n!} &= \frac{n \cdot n \cdot n}{n(n-1)(n-2)(n-3)\cdots 2 \cdot 1} \\ &< \frac{n \cdot n \cdot n}{n(n-1)(n-2)(n-3)} \leqslant \frac{8}{n}, \end{aligned}$$

if $n > 6$, it follows that $\lim\{n^3/n!\} = 0$.

9. a. Using the recursive equation one has

$$a_1 = 1, \quad a_2 = 3 - \tfrac{1}{1} = 2, \quad a_3 = 3 - \tfrac{1}{2} = \tfrac{5}{2} \quad \text{and} \quad a_4 = 3 - \tfrac{2}{5} = \tfrac{13}{5}.$$

b. From Part a it follows that $1 < a_1 < a_2 < 3$, and if k is any positive integer such that $1 \leqslant a_k < a_{k+1} < 3$ then

$$\tfrac{1}{3} < \frac{1}{a_{k+1}} < \frac{1}{a_k} \leqslant 1,$$

and so

$$1 < a_{k+1} = 3 - \frac{1}{a_k} < 3 - \frac{1}{a_{k+1}} = a_{k+2} < 3.$$

By the Principle of Mathematical Induction, it follows that $1 \leqslant a_n < a_{n+1} < 3$ for all positive integers n. This shows that the sequence $\{a_n\}_{n \geqslant 0}$ is increasing and bounded (below by 1 above by 3).

c. Since $\{a_n\}_{n \geqslant 1}$ is bounded and monotonic, the Monotonic Sequence Theorem implies that $\{a_n\}_{n \geqslant 1}$ is convergent.

d. If $\lim a_n = \ell$ (by Part c the limit is defined, and $1 \leqslant \ell \leqslant 3$) then also $\lim a_{n+1} = \ell$ also, and so

$$\ell = \lim a_{n+1} = \lim\{3 - 1/a_n\} = 3 - 1/\ell,$$

by elementary properties of the limit of a sequence. Therefore, $\ell^2 = 3\ell - 1$, or $\left(\ell - \frac{3}{2}\right)^2 = \frac{5}{4}$, and so $\ell = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ (since $\ell \geqslant 1$).

10. a. The series in question is a geometric series whose first term is $-\frac{2}{27}$ and whose ratio of successive terms is $-\frac{2}{3}$, which in absolute value is smaller than one. Therefore, the series in question converges and its sum is

$$\frac{-2/27}{1 - (-2/3)} = -\tfrac{2}{45}.$$

b. Since

$$a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2} = b_n - b_{n+1}, \quad \text{where} \quad b_n = \frac{1}{n} + \frac{1}{n+1},$$

one has $a_1 + \cdots a_n = b_1 - b_{n+1}$, and so

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = b_1 - \lim b_{n+1} = 1 + \tfrac{1}{2} - 0 = \tfrac{3}{2}.$$

11. a. Since

$$\lim \frac{-n^5 + n^2 + 1}{n^5 + 2} = \lim \frac{-1 + 1/n^3 + 1/n^5}{1 + 2/n^5} = -1$$

is not equal to zero, the series

$$\sum_{n=1}^{\infty} \frac{-n^5 + n^2 + 1}{n^5 + 2}$$

diverges by the Vanishing Criterion.

b. The series

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

is a divergent logarithmic p-series ($p = \frac{1}{2} \leqslant 1$).

c. If

$$a_k = \frac{k}{4k^3 - 5} \qquad \text{and} \qquad b_k = \frac{1}{k^2},$$

then $a_k, b_k > 0$ if $k \geqslant 2$ and

$$\lim \frac{a_k}{b_k} = \lim \frac{k^3}{4k^3 - 5} = \lim \frac{1}{4 - 5/k^3} = \tfrac{1}{4}.$$

Since $\sum b_k$ is a convergence p-series ($p = 2 > 1$), the series $\sum a_k$ converges by the Limit Comparison Test.

d. Since

$$0 < \frac{k!}{k^k} \leqslant \frac{2}{k^2} \qquad \text{if } k \geqslant 3, \qquad \text{the series} \qquad \sum_{k=1}^{\infty} \frac{k!}{k^k}$$

converges with the p-series $\sum k^{-2}$ ($p = 2 > 1$) by the Comparison Test.

Note: Alternatively, if $a_k = k!/k^k$, then $a_k > 0$ for $k \geqslant 1$ and

$$\lim \frac{a_{k+1}}{a_k} = \lim \left\{ \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} \right\} = \lim \left(1 + \frac{1}{k}\right)^{-k} = 1/e,$$

which is less than one, so the Ratio Test implies that the series $\sum a_k$ is convergent.

12. a. If $a_n = (-1)^n/(\arctan n)^n$ then

$$\lim \sqrt[n]{|a_n|} = \lim \frac{1}{\arctan n} = 2/\pi < 1,$$

so $\sum a_n$ is absolutely convergent by the Root Test.

b. Since

$$0 < \left| \frac{\sin m}{m^2 + m} \right| < \frac{1}{m^2}$$

and $\sum m^{-2}$ is a convergent p-series ($p = 2 > 1$), the series

$$\sum_{m=1}^{\infty} \frac{\sin m}{m^2 + m}$$

is absolutely convergent by the Comparison Test.

c. If

$$a_k = \frac{k^2 + 1}{k^3} = \frac{1}{k} + \frac{1}{k^3},$$

then $\sum a_k$ is divergent because it is the sum of a divergent p-series ($p = 1$) and a convergent p-series ($p = 3 > 1$). So $\sum (-1)^k a_k$ is not absolutely convergent. On the other hand, $\{a_k\}_{k \geqslant 1}$ is positive, decreasing (it is the sum of two decreasing sequences), and $\lim a_k = 0$, so $\sum (-1)^k a_k$ converges by the Alternating Series Test. Therefore, $\sum (-1)^k a_k$ is conditionally convergent.

13. a. Since $\{a_n\}_{n \geqslant 1}$ is positive, decreasing and

$$\lim\{a_n\} = \lim\left\{ a_n \cdot \frac{n}{n} \right\} = \lim\{na_n\} \cdot \lim\{1/n\} = 0 \cdot 0 = 0,$$

the series $\sum (-1)^{n-1} a_n$ is convergent by the Alternating Series Test.

b. If $a_1 = 1$, and $a_n = (n \log n)^{-1}$ for $n \geqslant 2$, then $\{a_n\}_{n \geqslant 1}$ is positive, decreasing and $\lim\{na_n\} = \lim\{(\log n)^{-1}\} = 0$, but $\sum a_n$ is a divergent logarithmic p-series ($p = 1$).

14. If

$$\alpha_n = \frac{e^n}{n^3}(x - 4)^n$$

then

$$\lim \left| \frac{\alpha_{n+1}}{\alpha_n} \right| = \lim \left\{ e\left(1 + \frac{1}{n}\right)^{-3} |x - 4| \right\} = e|x - 4|,$$

so by the Ratio Test the series $\sum \alpha_n$ is absolutely convergent if $e|x - 4| < 1$, *i.e.*, $4 - 1/e < x < 4 + 1/e$, and divergent if $x < 4 - 1/e$ or $x > 4 + 1/e$. If $x = 4 \pm 1/e$ then

$$\alpha_n = \frac{e^n}{n^3}(\pm 1/e)^n = \frac{(\pm 1)^n}{n^3},$$

and so $\sum |\alpha_n|$ is a convergent p-series ($p = 3 > 1$). Therefore, the interval of convergence of $\sum \alpha_n$ is $[\,4 - 1/e, 4 + 1/e\,]$ and the radius of convergence of $\sum \alpha_n$ is $1/e$.

15. One has

$$\begin{aligned}\frac{1}{(1+x)^3} &= (1+x)^{-3} = 1 + \sum_{k=1}^{\infty} \frac{(-3)(-4)(-5)\cdots(-3-k+1)}{k!} x^k \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 3\cdot 4\cdot 5\cdots(k+2)}{k!} x^k \\ &= \sum_{k=0}^{\infty} \tfrac{1}{2}(-1)^k (k+1)(k+2) x^k \\ &= 1 - 3x + 6x^2 - 10x^3 + \cdots.\end{aligned}$$

Alternatively, one has

$$\begin{aligned}\frac{1}{(1+x)^3} &= \tfrac{1}{2}\frac{d^2}{dx^2}\left\{\frac{1}{1+x}\right\} = \tfrac{1}{2}\frac{d^2}{dx^2}\sum_{n=0}^{\infty}(-1)^n x^n \\ &= \sum_{n=2}^{\infty} \tfrac{1}{2}(-1)^n n(n-1)x^{n-2},\end{aligned}$$

which is the same series as above (in which $n = k + 2$).